2-dimensional block cellular automata and conservation laws

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Abstract:

We consider 2-dimensional block cellular automata (BCA) with intention to meet simple physical requirements such as reversibility and space isotropy. Along with general study of the whole class of 2x2-neighborhood BCA, several rule subspaces that meet physical criteria are studied. It is also interesting to examine the possibility of the energy conservation expressed in terms of local rules. The energy function is set by the Ising model. Interpretation of domain, oscillating, particle and other patterns in terms of physical phenomena were studied. Some statements about impossibility of energy conservation laws expressed by 2D 2x2 BCA local rules were proved and verified experimentally with filtration methods.

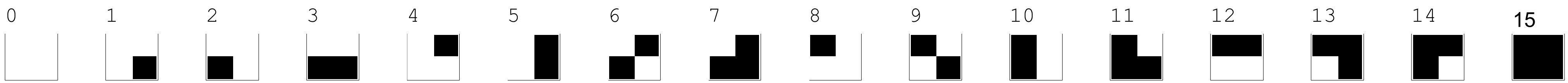
Results:

The Wolfram Demonstrations Project: designed an application to study 2D 2-color 2x2 BCA

- Proved a theorem of absence of non-trivial rules conserving the energy exactly
- Carried out filtration proving that there are no non-trivial rules conserving energy even approximately in reversible isotropic rule subspace
- Studied the reversible isotropic not flipping background rule subspace
- Studied interesting behavior of some rules of the above subspace
- Studied phase transitions in some of rules of non-reversible isotropic subspace

Block cellular automata. General rule space

The idea of block cellular automata (BCA) consists in replacing blocks of cells with blocks of the same size. At a particular step blocks do not overlap with each other. The blocks choice shifts every step. This project considers a square 2-color lattice where 2x2 blocks are replaced with other 2x2 blocks, then the block choice shifts in the diagonal direction by 1 cell. The boundary conditions are periodic. There are 16 types of 2-color 2x2 blocks:



So there are $16^{16} = 18446744073709551616$ rules in the general rule space of the 2-color 2x2 block 2D BCA. This rule space can be indexed by matching hexadecimal digit to every block like in the picture and then by transforming the obtained number to the decimal system. In this project the hexadecimal digit corresponding to the 0-th block stands at the leftmost position.

Physical requirements and rule subspaces.

- An physical requirement that is easy to meet in the BCA case (unlike in the ordinary CA case) is time reversibility. In the BCA case it means just that there are no similar blocks in the rule table (see image above), i.e. that the bottom row is a permutation of the top row. So there are 16! = 20922789888000 rules in the subspace of reversible rules.
- Another physical requirement is that uniform background cannot change (flip) its color.
- Physical space is usually supposed to be isotropic, so this requirement leads to the subspace of isotropic rules. If one rotates a block then its mapped image will also rotate for such rules. There are only 192 isotropic reversible rules that do not flip background.

Conservation laws. Ising model

Physical requirements can be also formulated by conservation laws. One of the most important conservation laws is the energy conservation representing the requirement of time homogeneity. Ising model considered here assumes black cells representing spins up (+1) and white cells spin down (-1) states. The energy of a lattice (if the external field is absent) is the sum over all the links, where each link between co-directional spins adds +1, and between anti-directional adds -1. One of the main purposes of this project was examination of possibility of energy law expressed in terms of local rules. Another conservation law that can be also interesting is the conservation of black cells, that represents conservation of the total magnetic moment in Ising model or a number of particles conservation in the lattice gas model. It is very easy to find the local rules that satisfy this law, because it means just that blocks transform to blocks with the same number of black cells.

The absence of rules conserving the energy

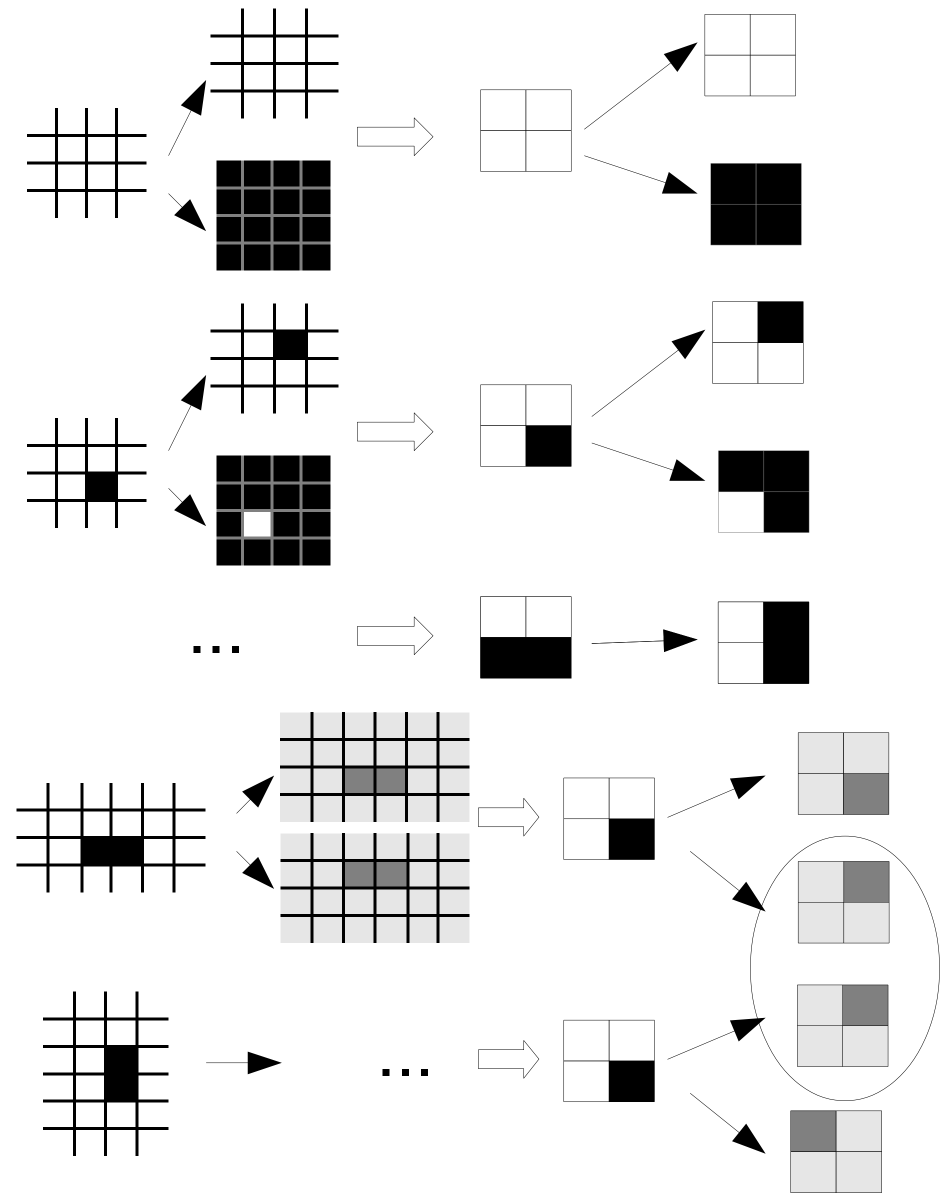
Theorem: there is no any nontrivial 2x2 2D BCA local rule, conserving the energy for all initial conditions even during one step.

Proof:

1) **White background.** To conserve the energy it can transform only to itself or to completely black lattice, because if there were cells of both colors there would be links that inrease the total energy. So a 2x2 white block can transform just to a black or white 2x2 block. Similarly for the black one.

2) **Only one block with one black cell on the white background**. A single cell on the background transforms into a single cell on the background. Since the background transforms to a background, a block with one cell on the background also transforms to a block with one cell on the background.

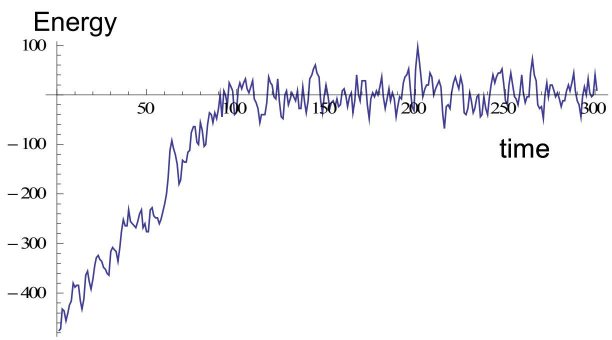
3) **Pairs of blocks on the background**. For example, two coupled cells in different blocks. To conserve the energy these cells have to remain coupled. By coupling cells in two ways (different boundaries of block) we are fixing the cell position and so one-cell block can transform only to itself or to the complement. Similarly for two-cell blocks. Only identical and complement transformation can conserve the energy.

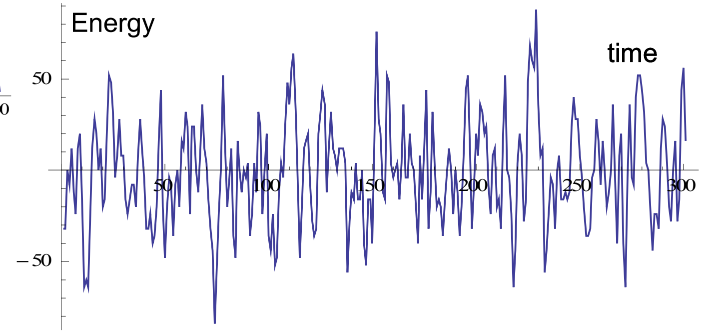


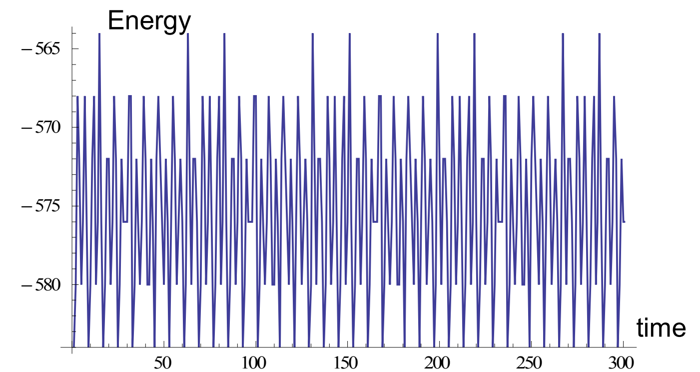
Note: *in sections 2) and 3) we assumed that there exists a background around the blocks. Yet if the lattice is so small that a block can be a neighbor to itself or a block can have the same neighbor on two sides, then 2) and 3) fail. Nevertheless, in such situation rules become not local, but system-wide.*

The absence of isotropic rules approximately conserving the energy

Since there are not any rules that conserve energy exactly, then the next point is to consider approximate energy conservation. First of all it is interesting in case of isotropic reversible not flipping background rules. (It is important to note here that the theorem proved above is true for all general rule space without any physical limitations.) The energy dependence on time has such typical forms:

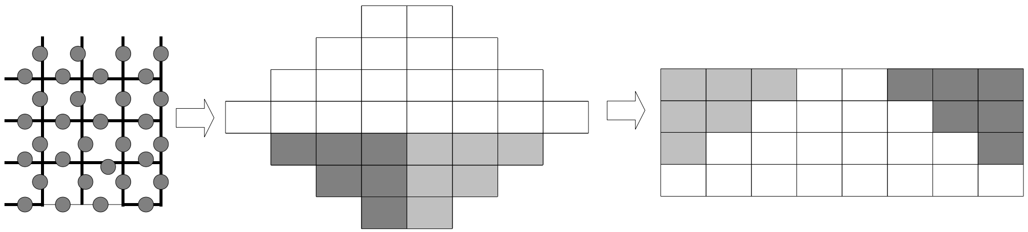






So it either thermolizes around a certain mean or is already thermolized or periodic. For energy conservation it would be fatal if thermolized means (or an ordinary mean in the periodic case) appeared to be the same. But we still could obtain energy conservation otherwise. To examine it this rule space was filtered in such a way: BCAs evaluated from small density (0.1), large density (0.9) and middle density (0.5), then difference between densities at the last step evaluated from small and large initials was compared with standard deviation of few latest steps of evolution from middle initials. If the difference were greater than deviation then the filter would return the rule. The filter gave some results, but they didn't appear in other runs of the filter (in those cases there were other results). So there no rules that could conserve the energy independently of initials have been obtained. We also carried out the search of rules where relative deviation in energy is less than in magnetic moment in random samples of all reversible rules subspace, but it has not given any results. This thermolizing behavior also prevents cluster formation or other interesting physical effects except the randomness formatiom in such systems.

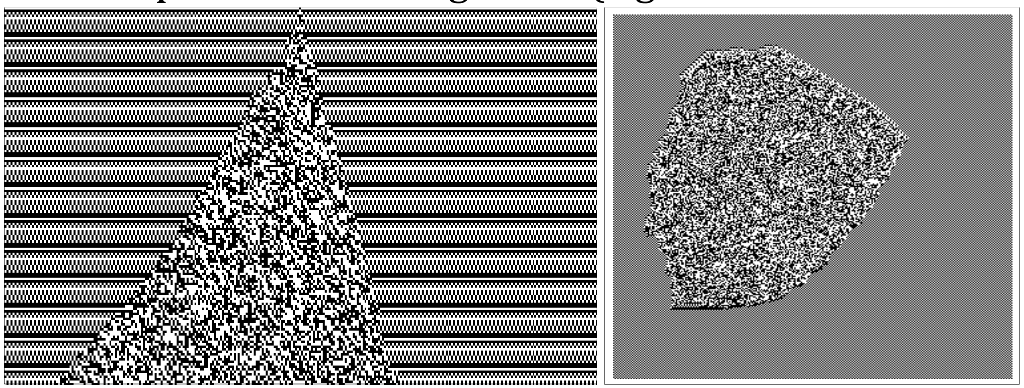
Another idea consists in considering the lattice of links instead of the spin one. This lattice can be done rectangular (see picture). The conservation of energy here is just a conservation number of blocks. But for all links lattices there exist spin lattices and the main problem is in conserving the acceptability during the evolution. And if one obtains the spin lattice only from a minimal number of links, then energy will not be conserved, even if black cells are conserved, but the links lattice is not acceptable.



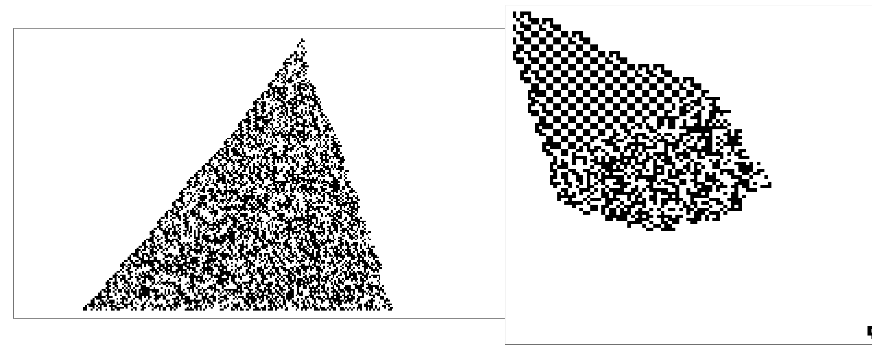
Note: boundary conditions are periodic only on the left and right sides and twisted on other sides.

Results of the reversible rules subspace studying

The subspaces of considered general BCA are very interesting by itself. The typical behavior of evolution from one black cell by non-isotropic reversible rules is increasing non-isotropic randomness (e.g. rule 14520071049273749310). Yet sometimes random mass in the center generates periodic structure of another phase than background (e.g. rule 10478226316561833375).

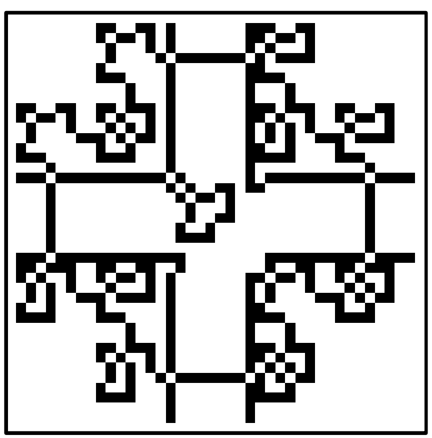


Rule 14520071049273749310



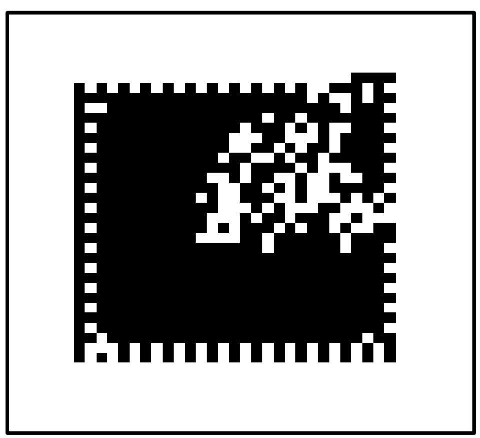
Rule 10478226316561833375

The behaviour of reversible not flipping background rules is more regular until the evolution does not reach a boundary.

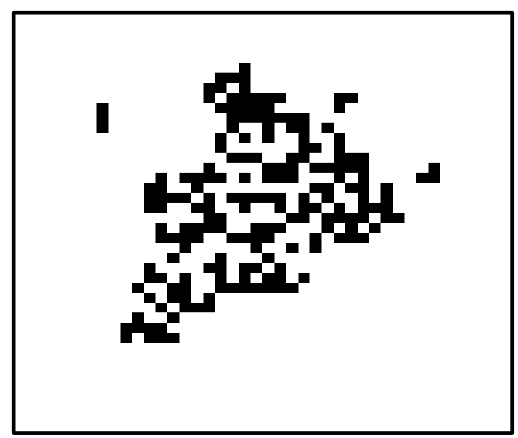


Rule 10683969969390006495
Regular behaviour (1 cell)

This rule has regular behaviour even after reaching the boundary.

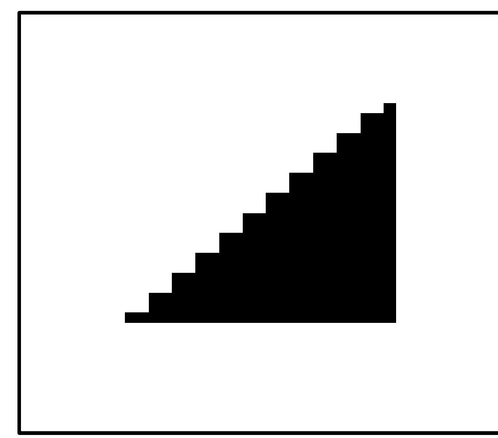


Rule 263551407201589215
Randomness in frame (1 cell)



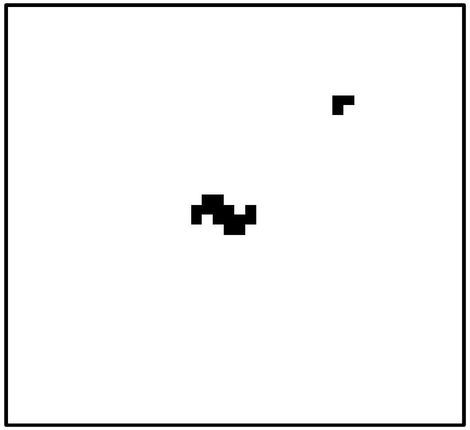
Rule 263551407201589215
Randomness (1 cell)

These two rules show random non-isometric behviour even in spite of their isometry.



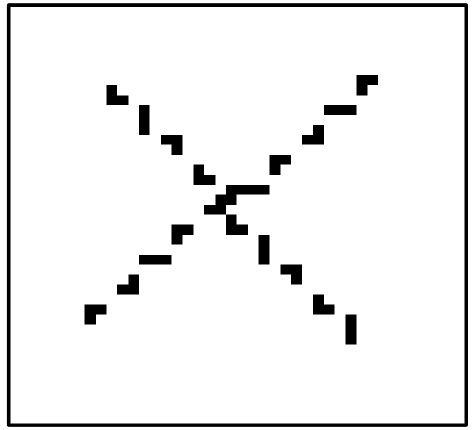
Rule 264401286749934735
Triangle (1 cell)

Even such simple behaviour (growing triangle) becomes random after reaching the boundary.



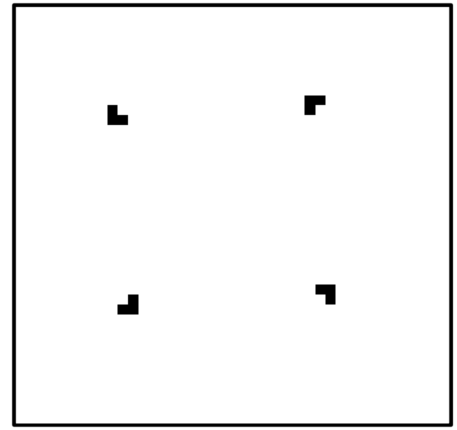
Rule 261583058097638895
One particle radiator (1 cell)

The rotating center radiated one particle.



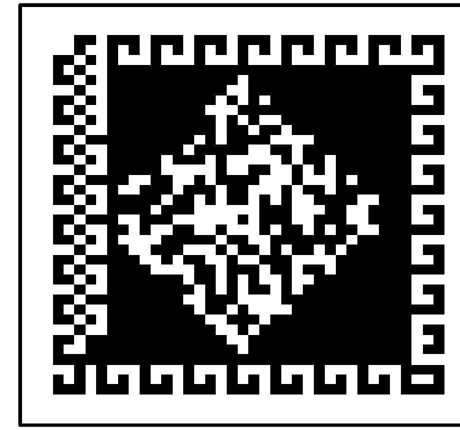
Rule 261583088162540415
Particle beams radiator (1 cell)

The center is radiating four particle beams.



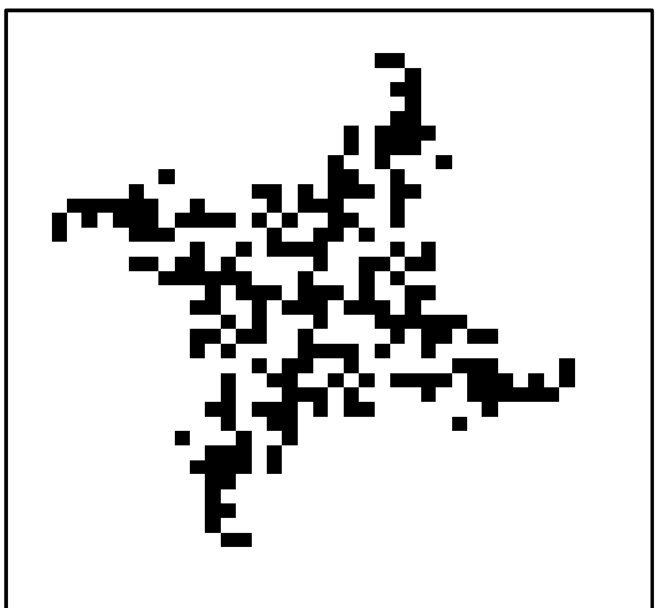
Rule 261583294270638975
Four particles (2x2 cells)

A 2x2 block decayed into four particles.

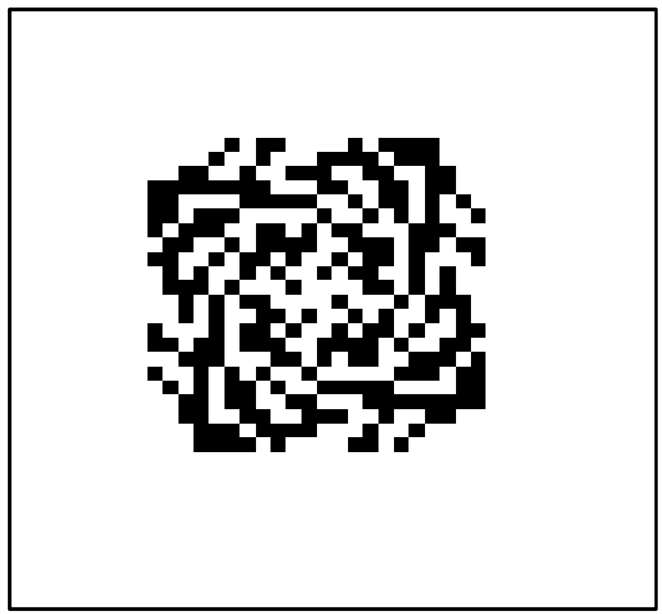


Rule 1068967613401572495
Random diamond in frame (1 cell)

Randomness in regular boundaries and regular frame

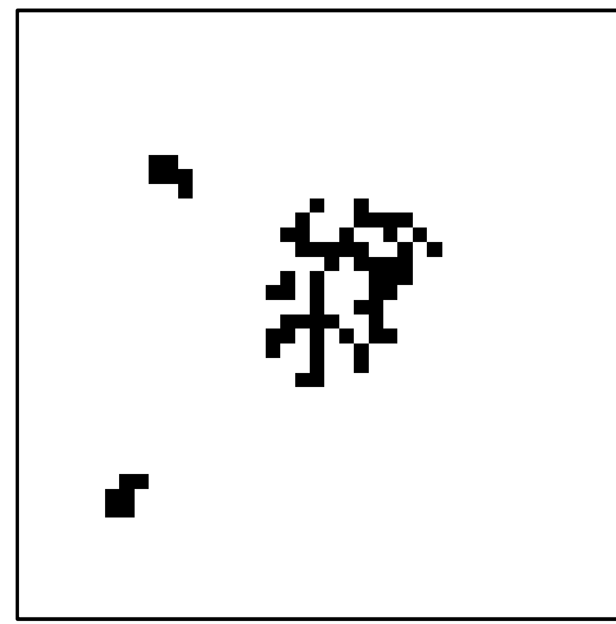


Rule 261861251725805535
Twisted star (2x2 cells)



Rule 261861260315283135
Rose (2x2 cells)

These two rules have complicated twisted regular behaviour.

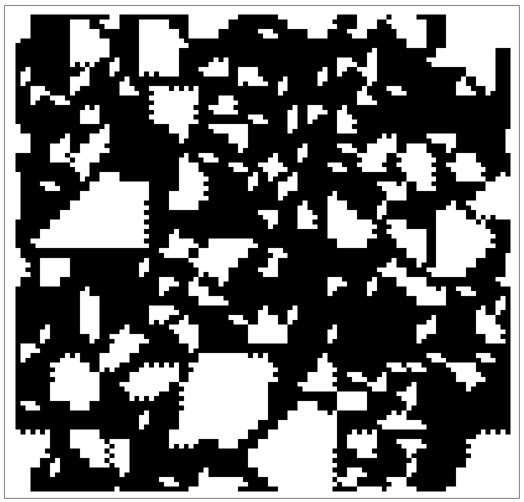


Rule 261861234545741295
Radiating mass (2 opposite cells in 3x3 block)

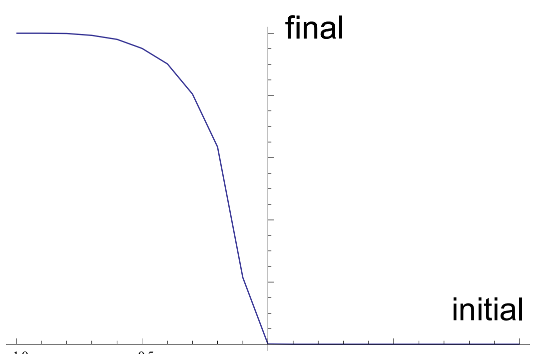
The random mass in the center radiates particles propagating faster than boundary of random area.

Irreversible rules. Clustering. Phase transitions

By using BCA irreversible rules it is very easy to present evolution to more and more energy-optimal state in local sense. To do it one can transform blocks with a small number of black cells (one black cell in considered 2x2 case) into a white block, and with a large number of black cells (three black cells) into a black block. We also assume that this does not flip the background. Other blocks can transform into different blocks, but one can also impose restriction of isometry. These rules depending on initial density lead the system to a one-colour state or to a cluster state. So there exists phase transition as a dependence of final density on initial density.



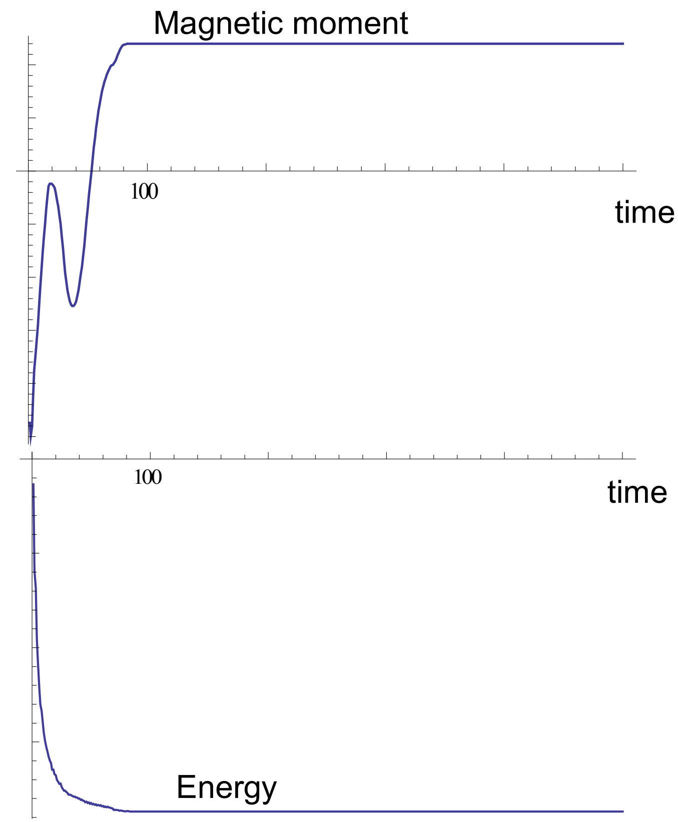
Rule 286968494657535
Frozen clusters



Phase transition for rule 286968494657535

Rule 2540967335796735
Drifting clusters (Phase separation

Rule 2540967335796735
Intermediate cluster state



Rule 2540967335796735
Magnetic moment and energy time dependence

In such a system we can see two phase transitions: from random to cluster state (energy drastically changes) and then to phase separated state (energy practically does not change).

Conclusions and further studies

The main negative result of this project is the theorem of absence of energy conserving rules. It is easy to see that this theorem is easily generalized onto the N-dimensional case. But in direct generalization onto cases 3x3x..., 4x4x..., ... N-dimensional blocks it only decreases a number of possible rules conserving the energy. Is it true that if blocks are not system-wide they cannot conserve the energy even at one step for all initials? Another interesting question: what is going on when the block is system-wide in only one direction or in a few, but not all directions. As it was mentioned in the section 3), the proof doesn't work in case of compactified dimensions.

It is also important to make the index of isotropic reversible not flipping background rules, that can be easily generalized on larger dimensions and block sizes (in this project the index was used just by the rule order in general index).

The obtained rules with particles and twisted structures deserve further consideration. Some interesting results of colliding such particles and their radiation have been already obtained.

Rules leading to phase transitions lie in subspaces of rules locally decreasing the energy, so this subspace needs to be studied and generalized. They seem to be easily obtainable.