Networks as Manifolds

Isabella Thiesen
Freie Universitaet Berlin

Abstract

The aim of this project is to identify the manifolds corresponding to networks that are generated by simple substitution rules from Stephen Wolfram's physics project. At first glance, some of the networks resemble a cell complex of a known surface. The idea is to provide techniques to substantiate this impression and figure out the geometric structure that might underly the purely combinatorially given network.

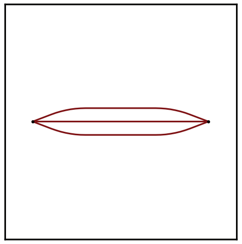
Results

For a given set of rules and the accordingly evolved networks we identified the underlying geometry. Apart from stationary networks and the real line, we obtained 2-dimensional surfaces, among these non-orientable, closed, curved and quotient surfaces.

Conclusion

We conjecture that network substitution systems are able to produce geometric structures. Combinatorics imposes several constraints on the topology of the surface, which in turn enables us to anticipate the geometry and identify manifold structures in evolutionary networks.

Introduction



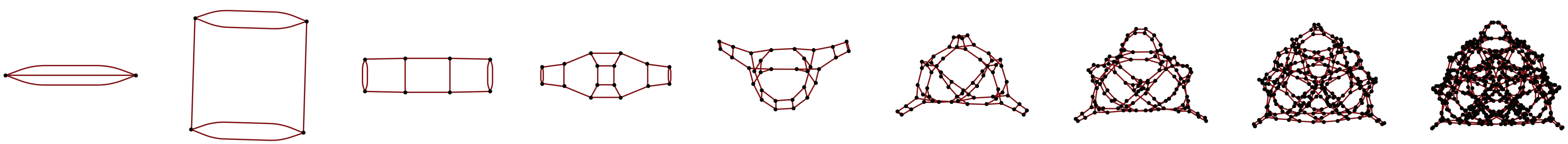
The starting point of our considerations is this very simple network, consisting of only two nodes, connected by three edges.

We obtain an orientation on the network by assigning a color (one of red, green and yellow) to each half-edge. This can be represented by a three-colored triangle at each node.

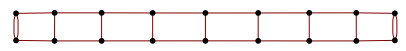
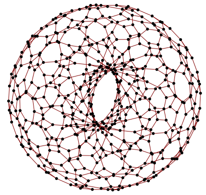
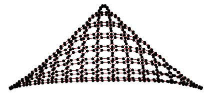
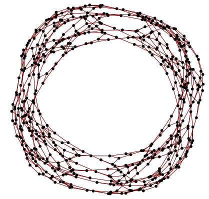
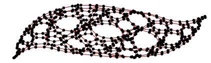
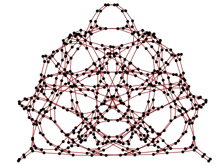
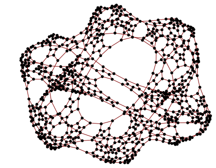
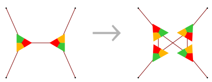
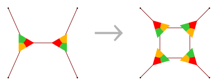
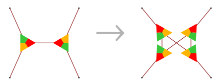
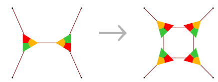
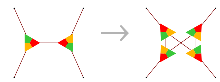
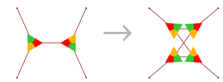
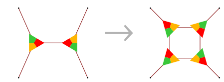
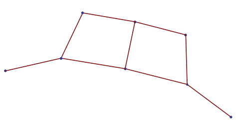
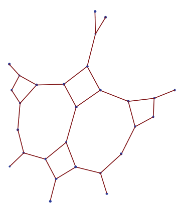
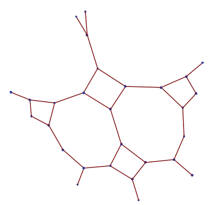
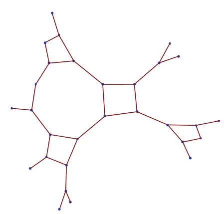
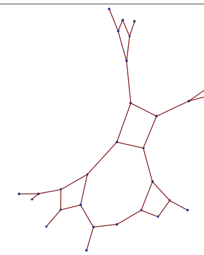
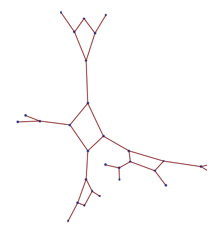
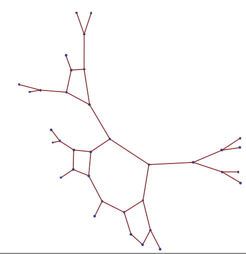
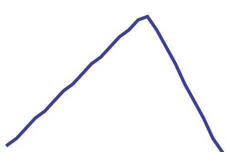
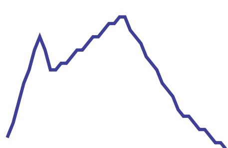
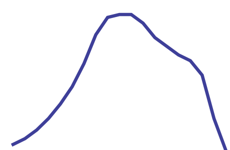
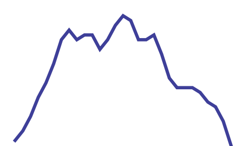
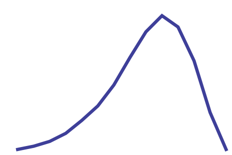
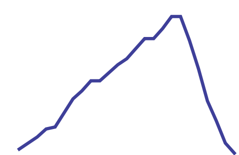
Applying network substitution systems to this network leads to a wide variety of networks, where the evolution is applied to each edge whose orientation corresponds to the one depicted in the rule.

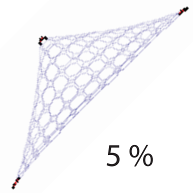
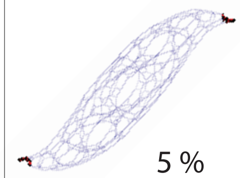
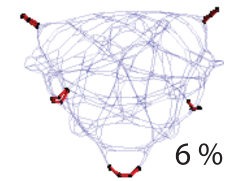
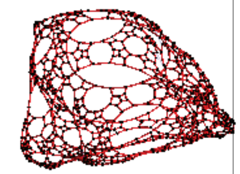
Although these networks are given purely in terms of combinatorics, we can think of imposing a distance measure by, e.g., taking the ordinary graph distance. We can then use this measure to consider linear plots of the network graphs, where we start at one node and plot the graph in ascending distance order of the nodes (see also the NKS book, p. 479).

The function describing the resulting pattern might then contain information about the dimension and type of structure underlying the network which in turn could relate to some well-known manifold.
For example, in the case of a network representing Euclidean n-space, the number of nodes out to distance r from a node should be O(r^n), such as the volume function behaves in the continuous case for n-dimensional balls. As seen in figure(f) on p.479, we can interpret graph functions that do not mimic this behaviour as curved.



Evolution of a network substitution system

Network							
Rule							
# of nodes after 8 steps	18	512	512	512	512	512	76 (here: 15 steps, 1711 nodes)
Local neighborhood							
Tiling up to singularities	—	{8,4}	{8,4}	?		{12,12,4} ?	?
Graph distance function							

Singularity plot	none	none	 5 %	none	 5 %	 6 %	 ? %
Name	Real line	Torus	Orbifold signature 442	?	?	?	Projective plane with holes?
Dimension	1	2	2	2	2	2	2
Curvature	0	0	0	?	?	-1?	1?
Construction	$\mathbb{R}$	$\mathbb{R}^2/p1$	$\mathbb{R}^2/p4$	?	?	?	$\mathbb{RP}^2 - \{D_i\}_{i=1,\ldots,n}$
Topology	χ=1	χ=0	χ=0	?	?	?	χ=1-n

Methods and tools to detect structures

At the very beginning we are left alone with a set of rules defining the network's edges. Fortunately, Mathematica gives us a first idea how this network might look like as a geometric object in space. The aim is now to discuss techniques that explain the underlying structure.

Graph distance function.

By studying the behaviour of the graph distance function we might detect inconsistencies which point to non-homogenuous points, i.e., singularities. The growth of the graph distance function acts as an indicator of the dimension of the network, where a growth of r^d-1 corresponds to a d-dimensional network.
For studying the local behaviour, it seems reasonable to look at nodes up to at least graph distance 3, as for less than 3 nodes the character of the node is difficult to decide (cf. constant curvature tilings). However, the distance considered should not exceed half the maximal graph distance.

Local neighbourhood.

For the singularity plot, we figured out the most common local neigbourhood up to distance 3 and marked deviating regions. This technique is especially useful when looking at quotient surfaces, as in this case inhomogenities can occur along the identified parts.

Geodesics.

By looking at geodesics, in particular longest ones, we obtain information about the metric. For the plot of the torus model below, we were looking for longest geodesics emanating from a point. We obtained a net of geodesics which indicates not only the symmetry of the object, but gives the network (and the plot) a shape-defining character.
For further work, we propose to compare (longest) geodesics on the network with those of the manifold and apply geodesics to construct faces.

Tiling.

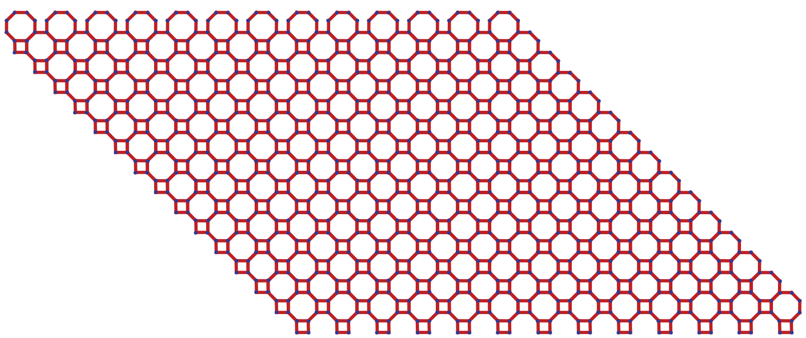
A uniform neighborhood, graph distance function or consistent geodesic behaviour can illustrate an identical network pattern at each of the nodes. This might indicate a tesselation of the covering space of the potential network geometry.

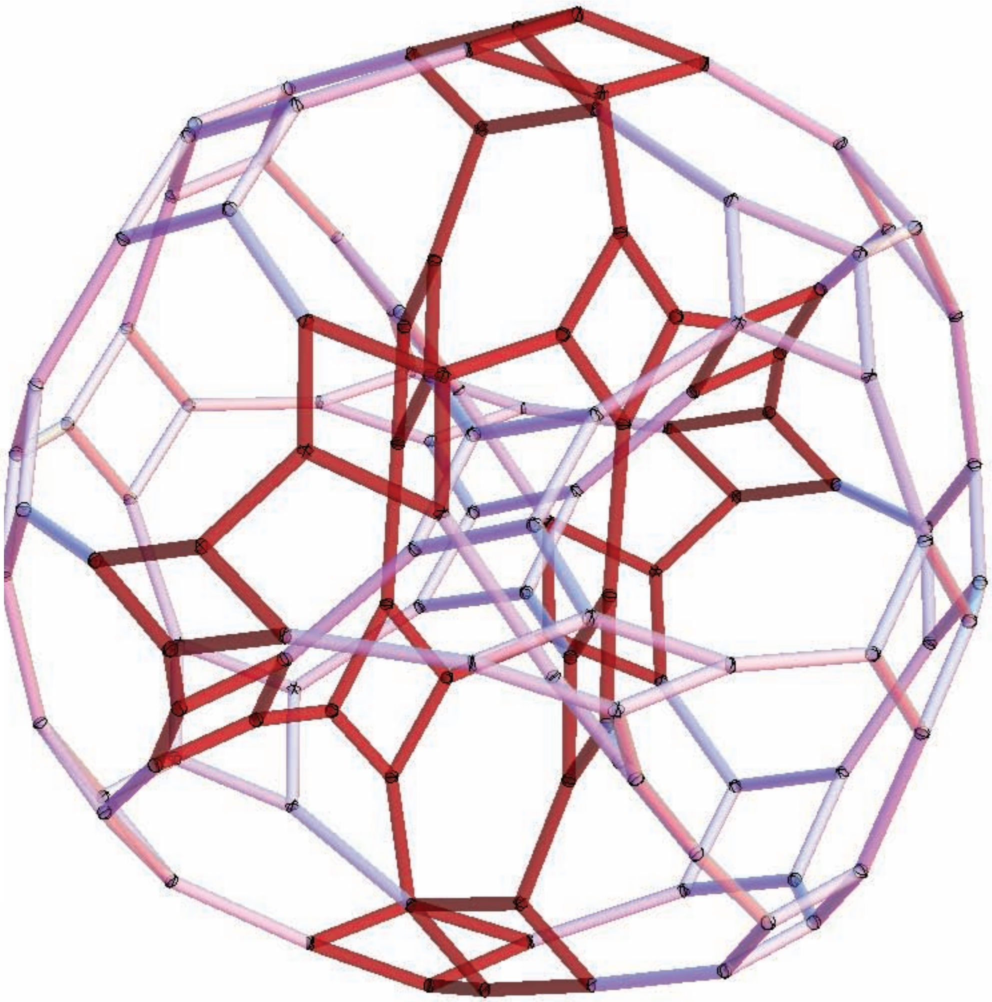
Structural overview of the toroidal network

Local neighborhood: Homogenuous →indicates a uniform and potentially closed surface.

Geodesics and tiling: By means of the geodesics emanating from each node and the local neighborhood we were able to figure out the three distinct loops based at each node, a 4-gon and two 8-gons. This indicates a quasi-regular {8,4} tesselation of the Euclidean plane.

Construction: The torus can be obtained as a quotient of the Euclidean plane by $\mathbb{Z}_2$ (or, in terms of the17 Wallpaper groups, p1). We can construct this explicit network by identifying opposite edges of an appropriate parallelogramm placed on a {8,4} tiling.

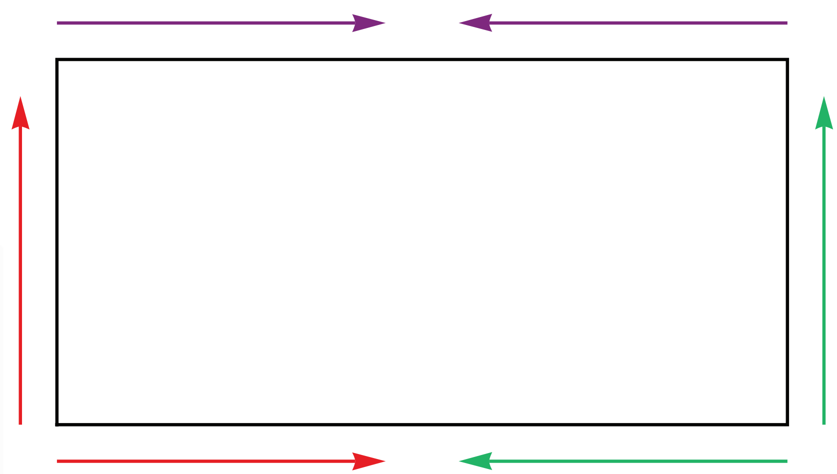




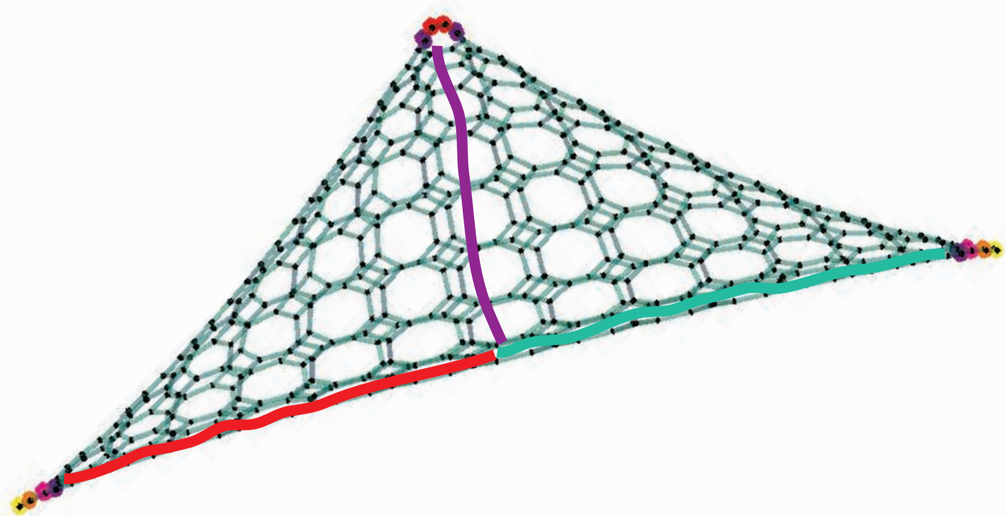
A closer look on the samosa orbifold

For the third model (which ressembles an Indian samosa), we obtained the same quasi-regular {8,4} pattern, apart from three singular regions. The geometry of the network is thus again Euclidean.

However, as we will discover next, its topology is spherical and therefore different from the previous torus.
Let us consider the uniform neighborhoods of the singularities. For the two sharper corners, the {8,4} pattern looping around it is missing three additional copies to make up a flat loop, and the bulge is missing one. The construction of the samosa can therefore be easily done by the following identification:



In fact, this network is a Euclidean orbifold, a generalization of a manifold whose covering is a manifold.
In terms of quotient surfaces we consider $\mathbb{E}^2/p4$, where p4 is the wallpaper group generated by two 90° rotational symmetries and one 180° rotational symmetry.



Conclusion

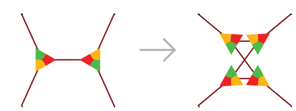
Simple rules can produce real manifolds!

Outlook and open questions

The probably most obvious question that is left open is the relation of rules and geometries. Is it possible to predict the evolution of networks for a specific rule? Is there a rule for every "constructable" surface?

Can we produce any model geometry with simple rules? Considering spherical and hyperbolic geometry, we conjecture that rule 6 and 7 in the table give us examples for positively resp. negatively curved networks. In the hyperbolic case, we assume the network to be a hyperbolic orbifold, recognizable by the uniform {12,12,4} hyperbolic tesselation apart from singular regions that might arise when identifying edges. We suppose the spherical network to be non-orientable and with boundary.

The torus and samosa model are part of the 17 compact Euclidean orbifolds that correspond to quotient surfaces of $\mathbb{E}$^2 by the 17 wallpaper groups. Can we obtain the other models as well by means of simple rules?



From another point of view, it is interesting to see if there are rules which produce networks that cannot be described by means of mathematical notation. The depicted network produced by the rule on the right might be a candidate for such a structure.