NOTES ON A MODEL OF TWO-PHASE FLOW IN POROUS MEDIA

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INTRODUCTION:

The understanding of the flow of fluids in porous media is one of the most challenging problems in continuum mechanics. Superficially, it is a very complicated problem becuase it seems that the details of fluid flow could well depend critically on the microscopic details of the medium in which flow is taking place. Thus, for example, sandstones and limestones might be expected to behave qualitatively differently. However, the known experimental data shows there is a universality in the type of behavior actually seen. One well known example of practical interest is the residual saturation of oil in a rock that as been flooded with water. A graph of the saturation versus the capillary number always shows the same qualitative behavior. (Figure 1). We would like to speculate that there is indeed some kind of universal behavior, of the kind found in equilibrium statistical mechanics, which is applicable to the issue of two-phase flow in porous media. If this conjecture is true, then drastically simplified models of two-phase flow will be able to make reasonable predictions when applied ot real rocks. These notes are devoted to the detailing of a specific type of model which we believe to be computationally tractable. In our model, the fundamental aim to describe two-phase flow in terms of the dynamics of blobs of oil immersed in a porous medium. Much work needs to be done, both experimental and theoretical, to determine whether this model is sensible. We believe the outlook to be promising.

Single-Phase Flow:

Single-phase flow in a porous medium is reasonably well understood[1]. In 1856, Darcy observed that the flow velocity was proportional to the pressure gradient in such a system. This is a natural extension of Poiseuille's law, except that there is no calculation of the coefficient of proportionality. There are three basic routes to finding that coefficient. The first is an analytic approach in which one models the porous medium by a collection of closely packed regular objects. In such systems, the Navier-Stokes equations can sometimes be solved. This approach can be criticized on the grounds that such a network for a porous medium is far too regular. Another approach is to model the porous medium by a series of spherical cavities (pores) joined together by cylindrical links (throats). In principle, the sizes of the pores and throats are completely adjustable (although pores are usually assumed to be about a hundred times larger than throats). Also, the way in which the pores and throats are linked together is, in principle, arbitrary. In practical computations, however, a regular lattice structure is usually used, although the dimensions of the pores and throats can be adjusted to any specified distribution. One then solves the Navier-Stokes equations in each pore and throat, and then patches together a global solution by appropriate adjustment of boundary conditions. Computational schemes based on the method described above are very difficult. However, in many situations, there is this system reduces to a simpler one related to electrical network theory[2]. In the simplest examples, pores are much bigger than throats, so they provide very little hydrodynamic resistance compared to throats. Throats are thus henceforth ignored. If Poiseuille's law applies in the throats, then the electrical analog is to substitute pores for nodes, voltage for pressure, current for flow velocity, and for a throat of radius r, length l, a resistor of value $R = \frac{\pi}{8} \frac{r^4}{\eta l}$ (η = viscosity). Networks of random resistors are very well understood, and reproduce the observed behavior of single-phase flow in rocks reasonably well, in both two and three dimensions.

Two-Phase Flow:

The techniques used for one-phase flow should equally well apply for two-phase flow. In practice, however, they turn out to be much more intractible. The reason is basically the two-component nature of the system. Oil in rock will exist initially as one large reservoir. If, however, water is forced into the region occupied by the oil, the oil-water interface will become very irregular. It will tend to split to give disjointed blobs of oil surrounded by water. These blobs may move, or fission, or collide and fuse with other blobs. The blobs are often observed to be very irregular. These effects tend to make a quantitative description extremely complex, unless one focuses only on the statistical properties of the system in question. If one follows the procedures developed for single-phase flow, one immediately discovers that analytic results, even for regular systems, simply do not exist. Simulations based on pores and throats are certainly possible, but complicated because of surface tension forces between the two phases, and because the interface can move around the network. Simulations of this in two-dimensions for lattices of size ~ 20 x 20 are now done,[3] but are very time consuming. They can be extended to three dimensions provided some improvements are made. There is an electrical analog, however, the oil-water interface now must be simulated by a non-linear device, whose position in a given circuit is time-dependent. For these reasons, we regard approaches based on detailed microscopic models of flow somewhat discouraging.

One model which does provide some useful information is the "percolation" model[4]. This applies in the limit of infinitesimal flow rates because then capillary pressure at the oil-water interface in the throats dominates over the externally applied pressure. In this limit, water will tend to move along the smallest throat in the oil-water interface until it finds a conductive path from one end of the sample to the other. In regular porous media, the interface thus turns out to uniformly advance through the system, whereas in random porous media, the interface becomes irregular, and leads to long finger-like trapped blobs of oil. This model has the advantages of being simple computationally. It shows scaling behavior which allows large numbers of pores and throats to be treated simultaneously. Furthermore, the model does show a degree of universality. One should, however, make the cautionary point that experiments are not yet sufficiently well-refined to either confirm or refute the predictions of this model.

What is required, therefore, is a simplified model, with which it is possible to do big simulations, and which will enable us to develop simple empirical laws which are valid in practical simulations. In our new model, we seek to achieve this by concentrating on how blobs of oil actually behave in a porous medium. Again, we should be somewhat concerned about the experimental situation. Whilst experimental studies have been done, it seems that it is still not clear how to describe two-phase flow and what the precise variables in the problem should be. Thus, it is not known whether or not the character of flow depends only on a small number of average properties of pores and throats or on the microscopic details of the medium in question.

In particular, for our model, it is necessary to attempt to get data on various possible elementary processes that a blob can undergo. It can, of course, flow with the water. If it does so, we do not know what shape it will take up. Blobs can clearly get stuck if they encounter a small throat, since then the capillary pressure at the interface is maximized. Having gotten stuck, is it possible for a blob to be liberated by a fluctuation in the pressure field? When do blobs fission or fuse? There seems to be little experimental data on these processes, we would like to encourage further work in this area to elucidate these phenomena.

We begin a description of our model then by finding when the "percolation" model is valid. If we consider a trapped blob surrounded by water, the percolation model can only be valid if the hydrostatic pressure across an average blob is much less than the capillary pressure trapping the blob. Consider a lattice of pores and throats of radii R and r respectively. The capillary pressure drop across a pore due to oil trapped in the throats Δp_c is given by

$$\Delta p_c \sim \frac{\gamma}{r}$$

where γ is the surface tension. There will also be a Poisenuille pressure drop of Δp_{ζ} in each throat given by

$$\Delta p_p \sim \frac{8\eta v R}{r^2}$$

where η is the viscosity of water, and v is the flow velocity in the throats. v can be related to the mean flow velocity of water in the sample u, by

$$v \sim (\frac{R}{r})^2 u$$

Thus, the "percolation" model is valid provided that $\Delta p_c >> \Delta p_p$ or

$$\frac{\eta u}{\gamma} << (\frac{r}{R})^3$$

The dimensionless quantity $C = \eta\mu/\gamma$ is called the capillary number. For a typical rock, $\gamma/R \sim 10^{-2}$ so that the "percolation" model is certainly not valid for $C > C_{crit} \sim 10^{-6}$. η_{crit} seems to correspond to the shoulder seen in Figure 1. Physically, one would expect that the "percolation" model fails once blobs of oil are mobilized by the flow of water. Again, qualitatively this does seem to correspond to the behavior shown in Figure one.

Our interest is in the prediction of the properties of two-phase flow for $C > C_{crit}$. For the reasons outlined above, we will now concentrate on the blobs themselves. Our aim is to first of all write down a master equation for the transport of blobs[6]. Our first question then is how to describe the blobs. Blobs are observed to appear in many different shapes, some rather fingered, some spherical. We will, at least temporarily, ignore this problem and classify blobs by their volume p; p should be regarded as a discrete variable corresponding to the number of pores filled by a given blob, and a velocity relative to the medium $\vec{v}$ at time t. The Baltzmann equation now reads

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f = Q$$

A represents the gains or losses to f_p as a result of interaction with the medium. One immediate difficulty is that, unlike a free particle, a blob will not move at a constant velocity. Thus, to proceed with this model, we must compute the forces exerted by water on a blob.

One way of calculating this is to recall the electrical analog of a porous medium. We can model a *fixed* blob in a porous medium filled with flowing water by a resistor network, exactly as in the single-phase case. The oil blob is represented by deleting these resistors corresponding to the location of the oil blob. This amounts to saying that there is no flow of water in the region occupied by oil. Since voltage is the analog of pressure, it is straightforward to compute the force acting on the blob. One should note that the lattice will destroy Galilean invariance, and so we do not necessarily know the force acting on a moving blob.

We have been able to obtain some data for the case of blobs, in two dimensions, which are thin rectangles, for regular lattices. The long side of the rectangle is perpendicular to the flow direction. (These were obtained by modification of a program to calculate single-phase flow. The nature of the program presented the calculation working for blobs of more interesting shapes, at least *pro tem*). On a 40 x 40 lattice of 1 Ω resistors, we deleted n resistors between the 20^{th} and 21^{st} rows. The voltage drop across the discontinuity is a measure of pressure. Hence, we can determine the force on the bar by summing up the voltage differences between the top and bottom of the bar. Table 1 summarizes our results. Figure 2 shows a graph of force vs. length. Figure 3 shows a graph of the pressure profile for the data in Table 1.

The best fit for the data is that the force is given by

$$F\alpha(\text{length})^{1.92484}$$

This method suggests a further approximation which seems to be valid when blobs are large compared to the pore separations. In this case, the discrete nature of the medium can be ignored and replaced by a conductivity sheet of conductivity σ. σ can be chosen to reproduce the average behavior of pores and throats. Thus, σ could be chosen to be constant to reproduce the behavior of the previous simulation, or chosen to be a stochastic variable so as to reproduce the behavior of real rocks. Thus, in the case of finding a force on an object, we simply solve Maxwell's equations with $n \cdot E = 0$ at the surface of the blob, and a constant electric field at infinity or radius R. We will solve this problem explicitly for a circular blob (see Figure 4). The situation is time-independent so

$$\begin{aligned} \text{div } B &= \text{div } E = 0 \\ \text{curl } B &= \sigma E \\ \text{curl } E &= 0 \end{aligned}$$

Thus, there exists a scalar potential Φ such that $E = -\nabla\Phi$, $\nabla^2\Phi = 0$ everywhere, and $n \cdot \nabla\Phi = 0$ on the circle. Expanding Φ in circular harmonics, and choosing $\Phi = 0$ at $\theta = \pi/2$

$$\Phi = E(r + \frac{R^2}{r})\cos\theta$$

where E is the value of the electric field at infinity, and (r,O) are polar coordinates whose origin is the center of the circle. The pressure field in the real problem is

$$p = -\frac{\eta v}{k}(r + \frac{R^2}{r})\cos\theta$$

where k is the permeability, and v is the flow velocity of the fluid at infinity.

Darcy's law should now be applied to find the flow velocity of this single-phase fluid in an effective porous medium. Since

$$\vec{u} = -\frac{k}{\eta}\vec{\nabla} p$$

the velocity field has component $u_{\parallel}$ parallel to the flow, and $u_{\perp}$ perpendicular to the flow

$$u_{\parallel} = v\left\{1 - \frac{R^2}{r^2}\cos 2\theta\right\}$$

$$u_{\perp} = -v\frac{R^2}{r^2}\sin 2\theta$$

Since the stress-energy tensor for a fluid is

$$\sigma_{ij} = -p\delta_{ij} + \eta(\nabla_i u_j + \nabla_j u_i)$$

the force on the sphere in the direction parallel to the flow is

$$F_x = 2\int_{-\pi/2}^{\pi/2} n^i \sigma_{ix} \cdot R d\theta$$

$$= 4v\eta \int_{-\pi/2}^{\pi/2} \left[\frac{R^2}{k}\cos^2\theta + (4\cos^2\theta - 2)\right] d\theta$$

$$= \frac{2\pi v \eta R^2}{k}$$

This result is qualitatively distinct from that obtained in Stokes' flow. In three dimensions, it would show that the force on a sphere in a porous medium was proportional to the volume or the sphere, whereas if the medium were absent, the force would be proportional to the radius of the sphere. This result is the same as would be obtained by the use of the Brinkmann equation. This result is also in qualitative agreement with our numerical simulation. This approach can also be used to calculate results for objects with different shapes. The reader must be cautioned that we have used a continuum description of the medium right up to the boundary of the blob. This is a little suspect, but future simulations using resistor networks will also allow us to determine the validity of that approximation.

One must also consider as part of Q various other elementary processes.

Fission:

In experiments, blobs are observed to fragment. It is not yet clear whether this is principally a property of the blob or the medium. In the continuum case, there are a number of interesting examples (rain). The fragmentation of blobs of water in air is poorly understood; the behavior of blobs of aniline in water (a popular executive toy) show extremely complex behavior; and the behavior of oil drops in vinegar is equally intriguing. For fluids in a porous medium, the situation is more complex. For example, the presence of ions in a rock will influence the behavior of a drop of ionic fluid. We feel that there are two basic ways a drop will fragment in a porous medium. One is that a blob can extend through a narrow throat. If there is a pressure gradient transverse to the throat, the blob will be ripped apart. If the blob is moving, it will be squeezed in two. Another is that a blob moving through a medium will leave a low pressure trail behind it (like an aeroplane). This will have the effect of making the near end of a blob tend to fall off. (This phenomenon was responsible for the spectacular demise of the prototype Tu-144 at the Paris air show). There are probably other processes which are important to fission. Clearer experiments will probably help to find the most important.

Fusion:

At first sight, one might think that blobs will coalesce if they collide. Thus, collision cross-sections are optical. However, this presumes there are no forces between blobs. The electrical analogy seems to suggest that there is a blob-blob repulsion, which is reasonably straightforward to compute. This computation should be done.

Sticking and Mobilization:

Blobs frequently seem to get stuck, due to capillary forces in small throats. Perhaps this means that stuck blobs should be treated separately in the transport equation, and, thus, distinguished from mobile blobs. This would seem to be reasonable given that a stuck blob needs a large viscous pressure gradient across it before it will become mobile. This also suggests that hysteresis effects will be quite important at moderate capillary numbers. This makes the problem rather messy.

Our aim is thus to clarify these effects, attempt to quantify them (perhaps only very approximately), incorporate them into the Q term in the Boltzmann equation, and design a statistical mechanics of blobs from that. This is an admittedly rather ambitious scheme, but, if universality applies to this class of problems, it is not a hopeless line of attack.

Prospects: We feel that there are two types of experimental input into this model.

The first is actual simulations of pores and throats in glass that enable flow to be explored. Many experiments have been done in the "percolation" regime, however, we feel that more experiments should be done at higher capillary numbers. We carried out two such experiments with Jing-Den Chen and believe that these support our idea that blobs should be treated as fundamental objects. However, films were then made of these experiments and their more leisurely viewing will probably help to refine some of the ideas presented here. Perhaps also future experiments could be performed starting with a single blob in the matrix, in an attempt to see if the single-blob equations are roughly correct. Another type of experimental input is from large-scale numerical simulations of the type mentioned earlier. These can be analyzed in great detail which enables one to find realistic forms for Q. Unfortunately, these simulations seem doomed in three dimensions with the present techniques. It is hoped that future developments will make such schemes practicable.

Finally, further calculations with our model are needed to see whether indeed the ideas presented here can in fact be turned into a tractable, simple mathematical scheme.

Table I. Estimated force on a Blob as the Number of Deleted Resistors Changes

Force	Number of Resistors Deleted
0.348	2
1.308	4
2.876	6
5.040	8
7.760	10
11.024	12
14.796	14
19.06	16
23.768	18
28.906	20

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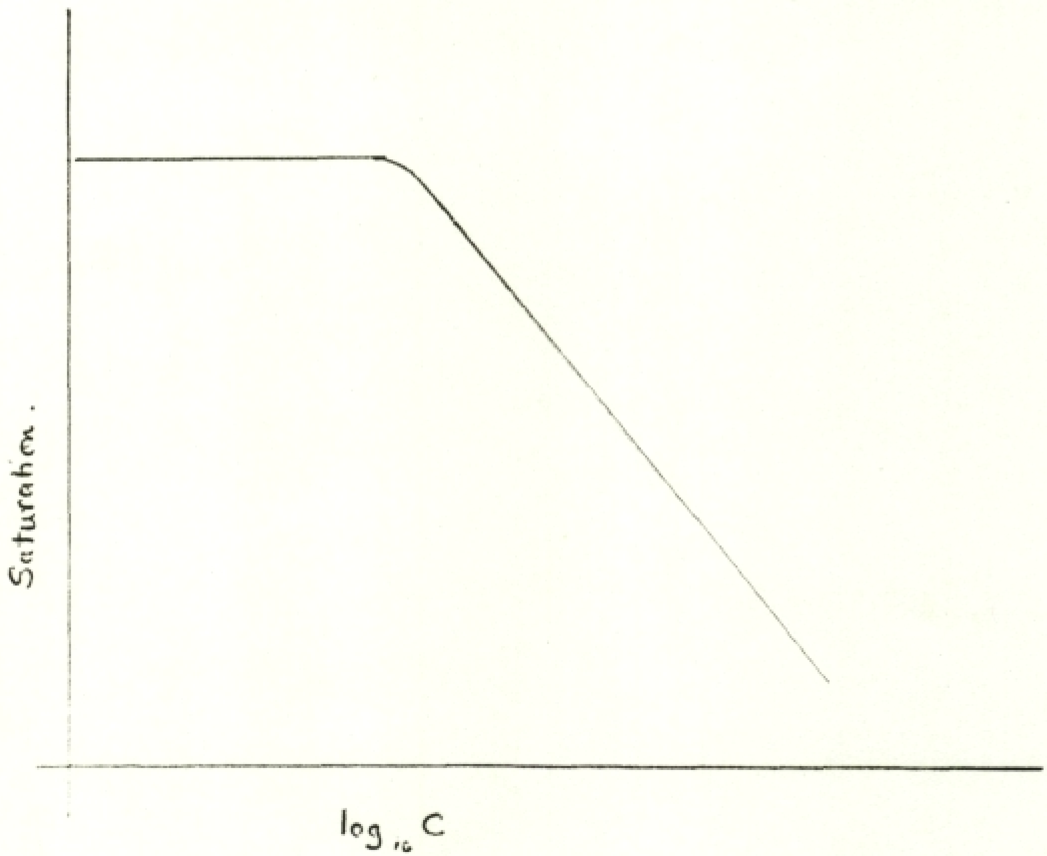

Figure 1

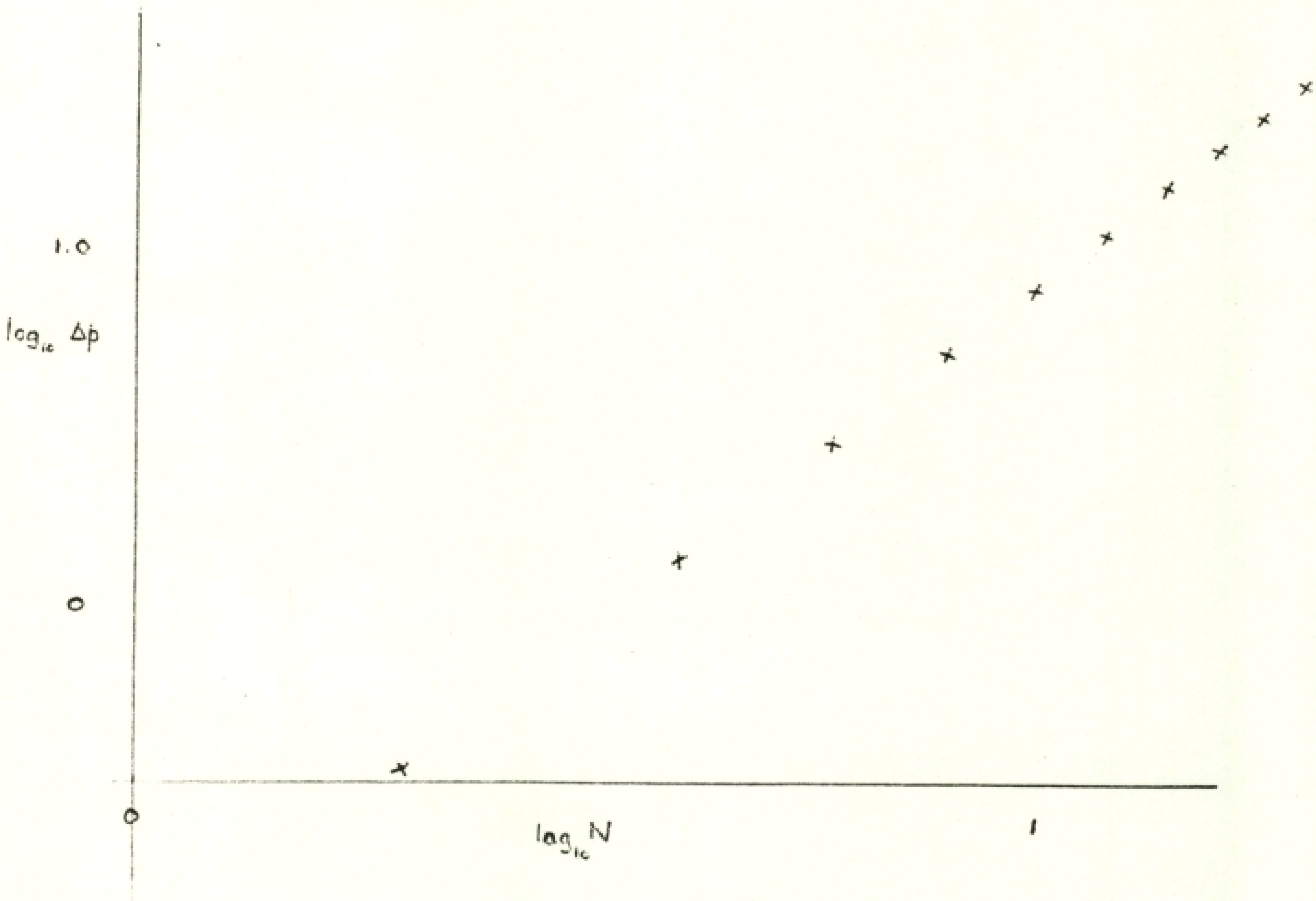

Figure 2

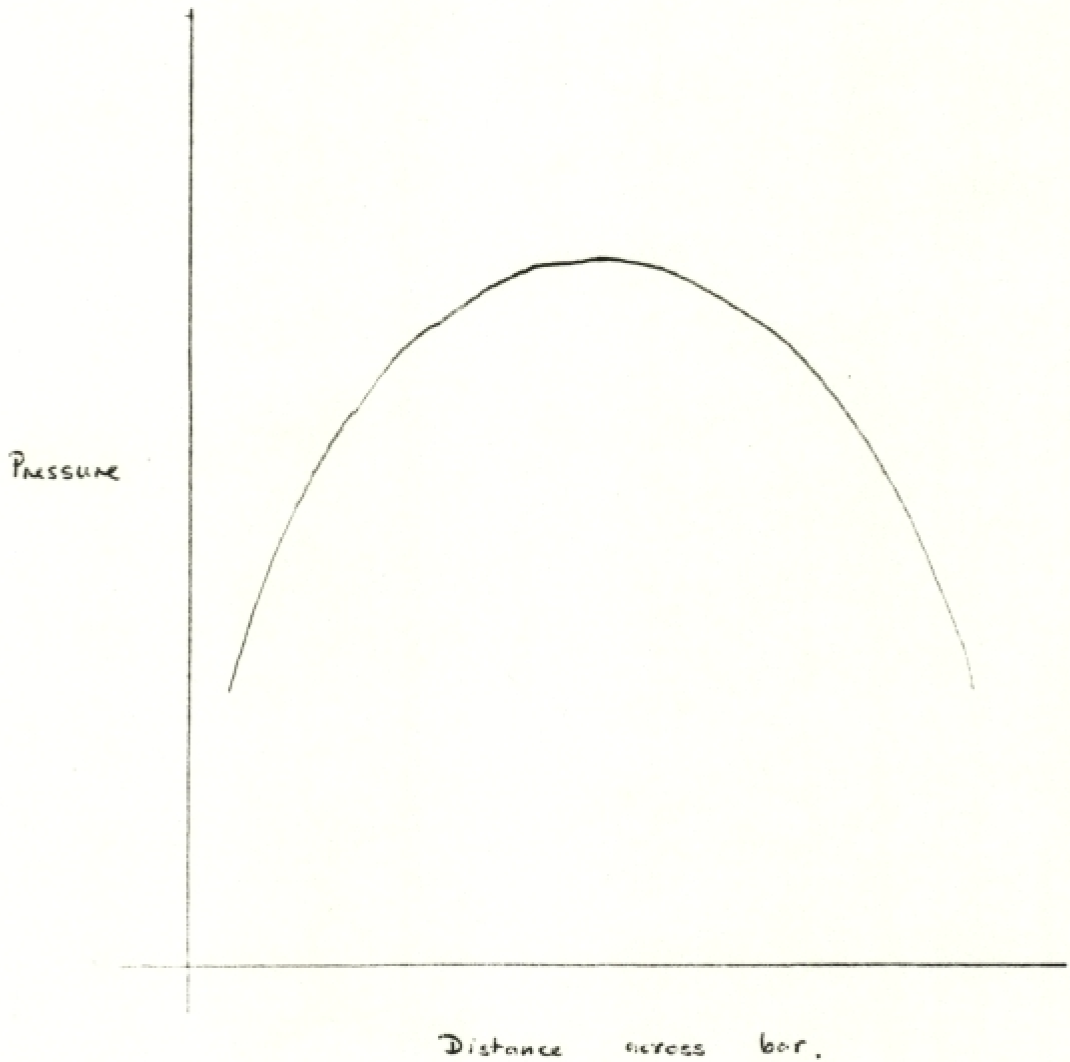

Figure 3

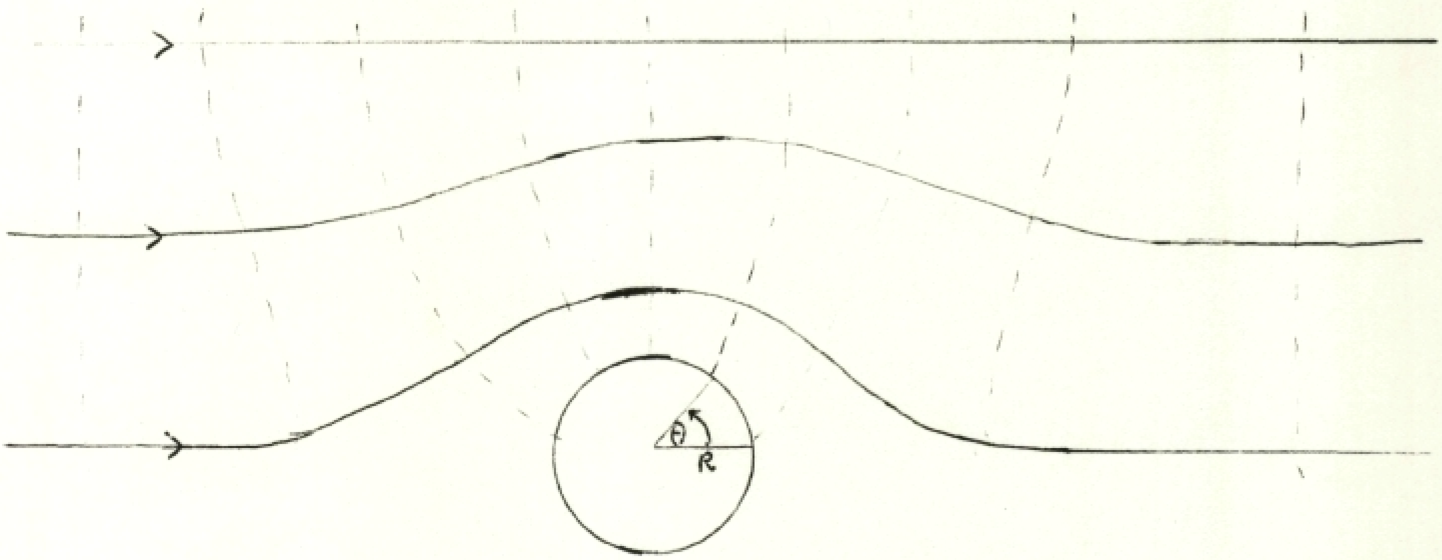

Figure 4

FIGURE CAPTIONS

1. A plot of residual saturation of oil in a typical rock against $\log_{10} C$. A shoulder always occurs at $C \sim 10^{-6} = C_{crit}$. For $C < C_{crit}$ the "percolation" model seems to be valid.

2. A plot of the logarithm of the of the force across the bar F against the logarithm of the length of the bar.

3. A graph of the pressure profile across the bar.

4. A sketch of electric field lines (solid) and equipotentials (dashed) for a conducting plate with a hole cut in it, representing a static blob in a porous medium with a fluid flowing past it.