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How Symbolic Summation Is Used in Particle Physics

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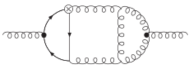
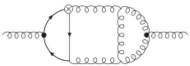
Der Wissenschaftsfonds.

Evaluation of Feynman Integrals



Behavior of particles

Evaluation of Feynman Integrals



Behavior of particles

$$\longrightarrow \quad \int \Phi(N, \epsilon, x)dx$$

Feynman integrals

$$\int_0^1 x\,dx = \text{?}$$

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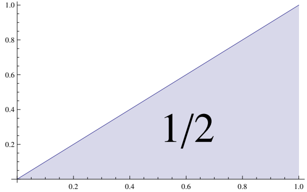
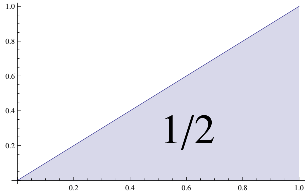
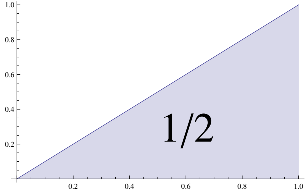


$$\int_0^1 x^1\,dx =$$



$$\int_0^1 x^2\,dx = \text{?}$$

$$\int_0^1 x^1\,dx =$$



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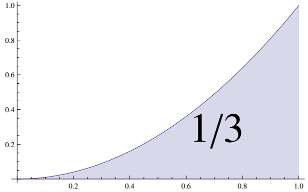

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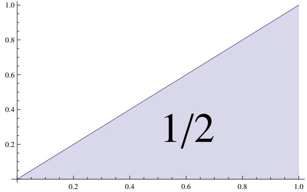

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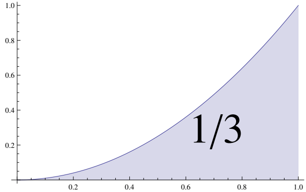
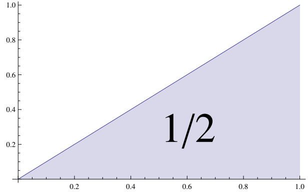


$$\int_0^1 x^3\,dx = ?$$

$$\int_0^1 x^1\,dx =$$



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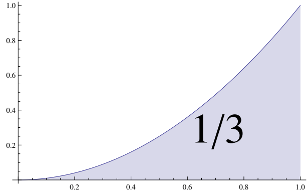
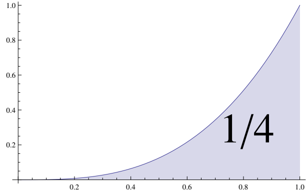


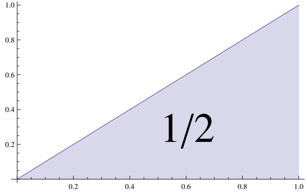
$$\int_0^1 x^3\,dx =$$



$$\boxed{\int_0^1 x^N\,dx = \frac{1}{N+1}}$$

für $N = 1, 2, 3, 4, \ldots$

$$\int_0^1 x^1\,dx =$$



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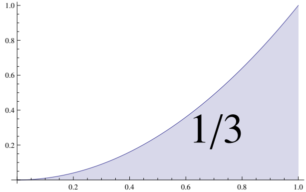

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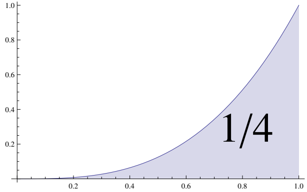


Feynman integrals

$$\int_0^1 x^N \, dx$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^N (1+x)^N}{(1-x)^{1+\varepsilon}}\, dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \dots dx_1 \, dx_2$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

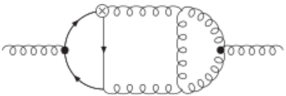
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^N(1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$\Vert$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

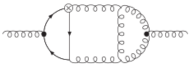
$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

Evaluation of Feynman Integrals



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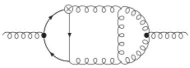
Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x) dx$$

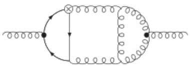
Feynman integrals

DESY
(J. Blümlein)

$$\sum f(N, \epsilon, k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

DESY

(J. Blümlein)

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

RISC

(Sigma-package)

expression in special functions

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)})$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \quad (= H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

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↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

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Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k)+(-n-1)S_1(n)-kS_1(k+n)-2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k)\Big]}$$

Zeilberger's creative telescoping paradigm

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Zeilberger's creative telescoping paradigm

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Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

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Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$||\qquad\qquad\qquad ||$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad -n\mathsf{A}(n)+(2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{A}} \frac{\mathbf{S[1,k]+S[1,n]-S[1,k+n]}}{\mathbf{kn(k+n+1)}};$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{A}} \frac{\mathbf{S[1,k]+S[1,n]-S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

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In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-\mathtt{n}\mathsf{SUM}[\mathtt{n}]+(1+\mathtt{n})(2+\mathtt{n})\mathsf{SUM}[\mathtt{n}+1] == \frac{(\mathtt{n}+1)\mathtt{S}[1,\mathtt{n}]+1}{(\mathtt{n}+1)^3}$

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In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{A} \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)};$$

Compute a recurrence

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In[4]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $$-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$$

Solve a recurrence

In[5]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, SUM[n]]}$

Out[5]= $$\left\{\left\{0, \frac{1}{n(n+1)}\right\}, \left\{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\right\}\right\}$$

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Compute a recurrence

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Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[5]= $$\left\{\left\{0, \frac{1}{\mathtt{n}(\mathtt{n}+1)}\right\}, \left\{1, \frac{\mathtt{S}[1,\mathtt{n}]^2+\sum_{\mathtt{i}=1}^{\mathtt{n}}\frac{1}{\mathtt{i}^2}}{2\mathtt{n}(\mathtt{n}+1)}\right\}\right\}$$

Combine the solutions

In[6]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[6]= $$\frac{\mathtt{S}[1,\mathtt{n}]^2+\sum_{\mathtt{i}=1}^{\mathtt{n}}\frac{1}{\mathtt{i}^2}}{2\mathtt{n}(\mathtt{n}+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(n,k,j)}\Big)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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2. Recurrence solving

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$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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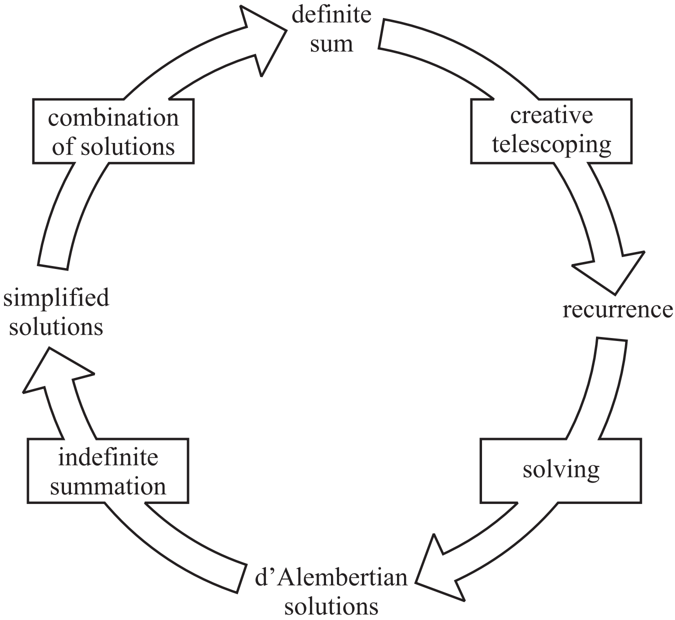
3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.


definite
sum
creative
telescoping
recurrence
solving
d'Alembertian
solutions
indefinite
summation
simplified
solutions
combination
of solutions

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\|$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

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$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

In[17]:= $<<$ **Sigma.m**

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In[18]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[19]:= $<<$ **EvaluateMultiSums.m**

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In[17]:= $<<$ **Sigma.m**

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EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[20]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[21]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

In[17]:= $<<$ **Sigma.m**

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In[18]:= $<<$ **HarmonicSums.m**

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EvaluateMultiSums by Carsten Schneider © RISC-Linz

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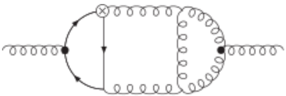
In[21]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[21]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

Example 1:

massive 3-loop ladder integrals

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1} \qquad ||$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

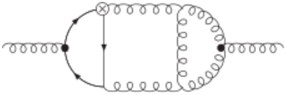
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

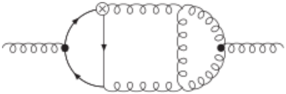
$$+ [x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k \Big]$$

$$\times (1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times [x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \big(- \ldots 1)^N(2N+1) / \ldots N(N+1) - \frac{13}{N}\big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ (2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32{\color{red}S_{-2,1,1}(N)} + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right){\color{red}\zeta(2)}$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(- \ldots 1)^N(2N+1) \over N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \Big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 \ldots} \ldots 3 \ldots)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4 \ldots)^N \ldots 1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

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$$+ \frac{4(3N - \ldots}{N(N + \ldots} \ldots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\big)$$

$$+ \big(\frac{(-1)^{\ldots}}{N(\ldots} \ldots (N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

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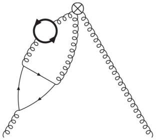
$$S_{-2,1,1}(N) = \sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

Example 2:

2-mass 3-loop Feynman integrals

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



All diagrams are produced with axodraw (J. Vermaseren).

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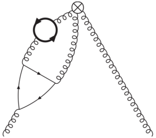

Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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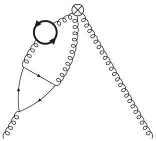

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Typical triple sum:

$$\sum_{j=0}^{N}\sum_{i=0}^{j}\sum_{k=0}^{i}\frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon}\times 2^{-2+\varepsilon}e^{-\frac{3\varepsilon\gamma}{2}}\eta^{k}\times$$

$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

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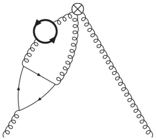

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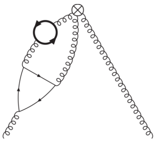
$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

6 hours for this sum

$\sim$ 10 years of calculation time for full expression

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



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expression (377 MB) consisting of 8 multi-sums

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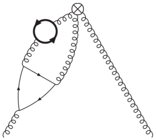

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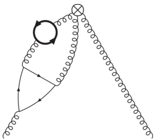
Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

sum	size of sum (with ε)	summand size of constant term	time of calculation		number of indef. sums
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{\infty}$	17.7 MB	266.3 MB	177529 s	(2.1 days)	1188
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{\infty}$	232 MB	1646.4 MB	980756 s	(11.4 days)	747
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{\infty}$	67.7 MB	458 MB	524485 s	(6.1 days)	557
$\sum_{i_1=0}^{\infty}$	38.2 MB	90.5 MB	689100 s	(8.0 days)	44
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}$	1.3 MB	6.5 MB	305718 s	(3.5 days)	1933
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2}$	11.6 MB	32.4 MB	710576 s	(8.2 days)	621
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{i_2}$	4.5 MB	5.5 MB	435640 s	(5.0 days)	536
$\sum_{i_1=3}^{N-4}$	0.7 MB	1.3 MB	9017s	(2.5 hours)	68

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



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expression (377 MB) consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

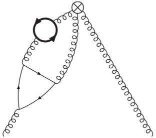
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i}}{i\binom{2i}{i}} \right) \left(\sum_{i=1}^{h} \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i} \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{(1-\eta)^k}{k}}{j}}{i\binom{2i}{i}} \right).$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies →

expression (95 MB) with
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- 1000 double sums
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↓ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

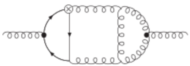
↓ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

← Sigma.m (32 days)

expression (8.3 MB) consisting of 74 indefinite sums

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

DESY
(J. Blümlein)

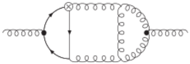
$$\sum f(N, \epsilon, k)$$

complicated
multi-sums

RISC
(Sigma-package)

expression in
special functions

Evaluation of Feynman Integrals



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DESY
(J. Blümlein)

LHC at CERN

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expression in
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applicable

Challenges

More than 10000 difficult Feynman integrale have been cracked

(some needed more than 50 days of calculation time)

$\downarrow$

over **1 million multiple-sums** have been simplified

(in average triple-sums)

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Tools/equipement

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  with third party funds: J. Ablinger, A. Behring, A. de Freitas, A. Hasselhuhn, C. Raab, M. Round, K. Schönwald

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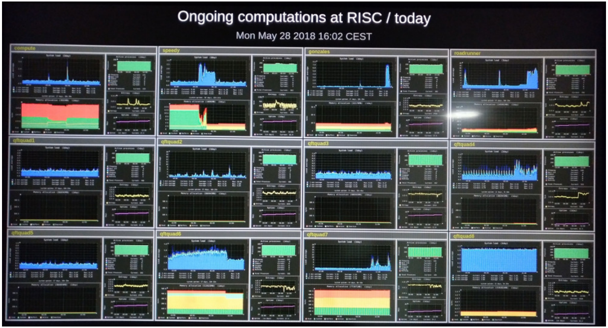
- DESY donated **8 strong compute servers** to RISC

qftquad1–qftquad8:

4.5 TB RAM and 112 TB fast disk memory



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- **new efficient Mathematica packages for the mass production**

Our software packages (in Mathematica):

1. the `Sigma` difference ring package of CS for symbolic summation.
2. the `HarmonicSums` package of J. Ablinger for special functions
3. the `EvaluateMultiSums` package of CS
4. the `FSums` package von F. Stan for symbolic summation.
5. the ρ`Sum` Package of M. Round for holonomic summation.
6. the `MultiIntegrate` package J. Ablinger
7. the `SumProduction` Paket CS for compactifying huge sum expressions.
8. the `SolveCoupledSystem` package of CS for solving coupled systems of differential equations.
9. the `ANCONT` package of J. Blümlein for numerical evaluation.