

# History of Mathematics Project: Learning Journeys for Kids and Others

Bernat Espigule, Wolfram Research

   #WolframTechConf

@bernatree



# Website Organization

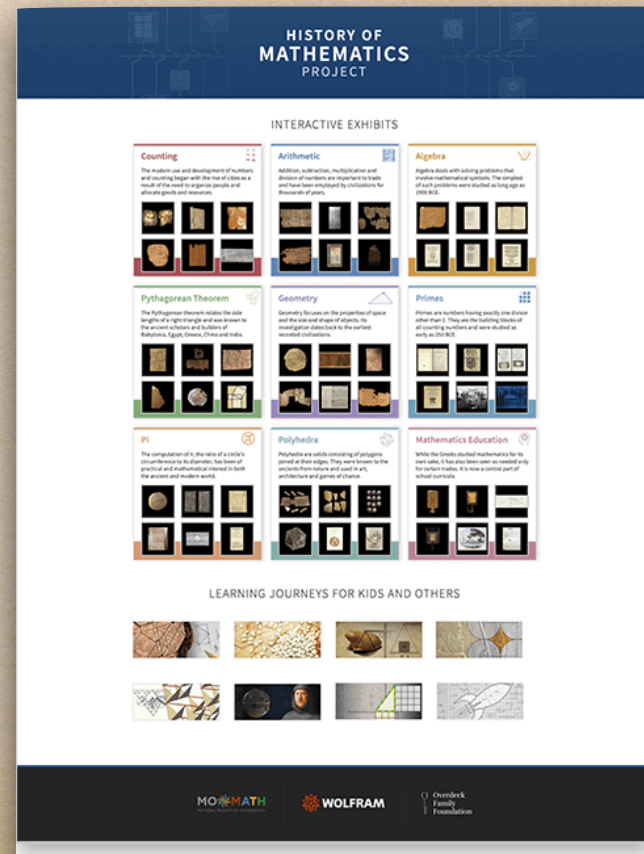
- ◆ Home page ([history-of-mathematics.org](http://history-of-mathematics.org))
- ◆ 9 virtual exhibits
- ◆ 74 artifact pages
- ◆ 8 learning journeys



Ancient Games of Chance



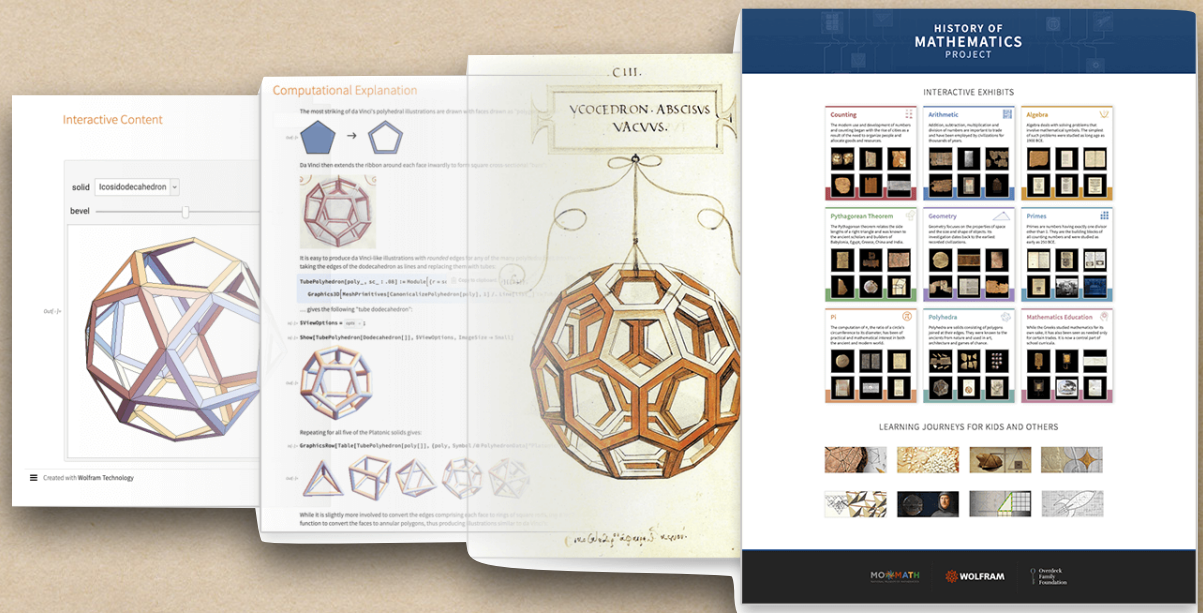
Balancing Ducks, Frogs and Grasshoppers





# The History of Mathematics Development Team

- ◆ Andrea Gerlach
- ◆ Eric Weisstein
- ◆ Bernat Espigulé
- ◆ Sarah Keim Williams
- ◆ Lorí Goodman
- ◆ with additional contributions from 50+ domain experts in relevant areas of the history of mathematics, notation, and the study of antiquities





# 9 Interactive Exhibits + 8 Learning Journeys

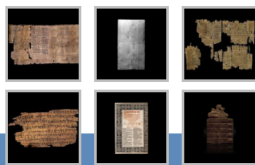
## Counting

The modern use and development of numbers and counting began with the rise of cities as a result of the need to organize people and allocate goods and resources.



## Arithmetic

Addition, subtraction, multiplication and division of numbers are important to trade and have been employed by civilizations for thousands of years.



## Algebra

Algebra deals with solving problems that involve mathematical symbols. The simplest of such problems were studied as long ago as 1900 BCE.



## Pythagorean Theorem

The Pythagorean theorem relates the side lengths of a right triangle and was known to the ancient scholars and builders of Babylonia, Egypt, Greece, China and India.



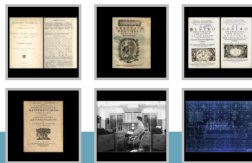
## Geometry

Geometry focuses on the properties of space and the size and shape of objects. Its investigation dates back to the earliest recorded civilizations.



## Primes

Primes are numbers having exactly one divisor other than 1. They are the building blocks of all counting numbers and were studied as early as 250 BCE.



## Pi

The computation of  $\pi$ , the ratio of a circle's circumference to its diameter, has been of practical and mathematical interest in both the ancient and modern world.



## Polyhedra

Polyhedra are solids consisting of polygons joined at their edges. They were known to the ancients from nature and used in art, architecture and games of chance.



## Mathematics Education

While the Greeks studied mathematics for its own sake, it has also been seen as needed only for certain trades. It is now a central part of school curricula.



Mathematical Beans and Knotted Strings



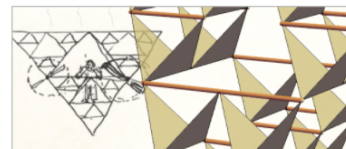
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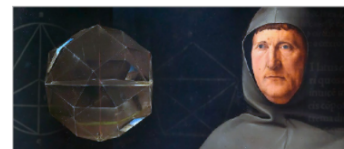
Show Your Work!



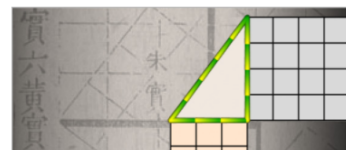
Squaring the Apsamikku Circle



Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles




Ancient Games of Chance



# 9 Interactive Exhibits


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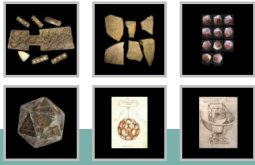
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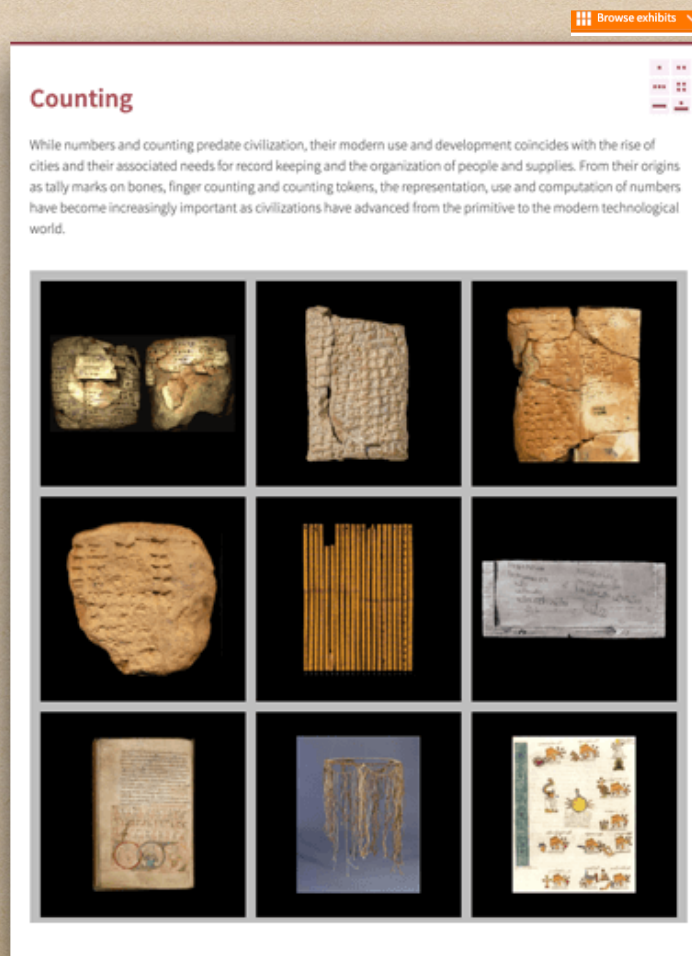
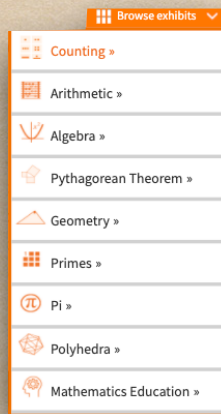


- ◆ Counting
- ◆ Arithmetic
- ◆ Algebra
- ◆ Pythagorean Theorem
- ◆ Geometry
- ◆ Primes
- ◆ Pi
- ◆ Polyhedra
- ◆ Mathematics Education



# Each Interactive Exhibit Page Contains:

- ◆ Navigation to other exhibits
- ◆ Short description
- ◆ Clickable timeline
- ◆ Clickable thumbnails for several artifacts









## Each Math Artifact Page Contains:

[illegible][illegible]

Figure 1 illustrates the game of Go. The top part shows a sequence of moves: a pair of stones (black and white) being captured, a pair of stones being moved, and a black stone being captured. Below this, a Go board is shown with a 'Starting State' and an 'Ending State' after a sequence of moves. The board is labeled 'YIPPAH BOLE' and 'YIPPAH BOLE'.

## Other Resources

### Additional Reading

[illegible]

### Additional Links

- [Dumbarton Oaks Research Library and Collection: The Role of Knots](#)
- [The Guaman Poma Website: A Digital Research Center of the Royal Library, Copenhagen, Denmark](#)
- [MALI, Museo de Arte de Lima: Yaguna- inca style 1400 AD - 1532 AD](#)
- [Metropolitan Museum of Art: Gama Board \(Yupana\)](#)
- [Museo Chileno de Arte Precolombino: Yagunas and pallasas: Stories to calculate](#)
- [Wikipedia: Yupana](#)
- [YouTube: Yaguna Inka - Competencia Matemática Tawari Pachay en Mito del Andino](#)











## Each Math Artifact Page Contains:

[illegible]



## Each Math Artifact Page Contains:

[illegible]



# Each Math Artifact Page Contains:

**Interactive Content**

In the abacus recording device of the Andean peoples, knots were used to indicate numeric values based on their positions and types, like the positions of each of the squares on a yupana board indicated numeric value. Puma de Ayala's yupana board consists of five groups of squares which groups indicate tens, hundreds, thousands, ten and one, respectively, within each group, there are four squares represent the values 1, 2, 3 and 4.

The value being represented on a yupana is recorded by placing one, two or more markers on the squares. Poma's (1605) and Poma's (1625) suggest the following arrangement of markers for the integers between 1 and 9:

0	1	2	3	4	5	6	7	8	9
	●	●●	●●●	●●●●	●●●●●	●●●●●●	●●●●●●●	●●●●●●●●	●●●●●●●●●

Using these arrangements, one can represent integers (as long as the number of available columns) can be represented on the board by using differently colored markers, more than one marker can be represented at the same time.

Representation of multiple numbers can be performed through the mechanical operations of erasing, moving and restoring the markers.

**1615**  
**Incan Yupana**  
*Ancient Incan abacus*

Yupanas were ancient calculating device of the Incas. The term yupana derives from the indigenous Andean Quechua language in which yupana means "to count". The most important historical document concerning the yupana is a sketch made by Felipe Guamán Poma de Ayala in 1615, which was originally lost then rediscovered at the Royal Library of Copenhagen in 1916. While the yupana described by Poma de Ayala is lost and is a 4 x 4 grid, examples shaped as polygons with different numbers of sides.

The earliest known yupana have been found in Peru, consisting of a grid of squares, some of which are carved into a grid by the Andean peoples. However, the type of yupana described by Poma and interpreted to perform calculations. While there are many examples of yupana, the type of yupana described by Poma and interpreted to perform calculations. While there are many examples of yupana, the type of yupana described by Poma and interpreted to perform calculations.

**Artifact format**  
Unknown

**Artifact origin**  
Modesto Omiste, Potosí, Bolivia

**Current artifact location**  
Museo de Instrumentos Musicales de Bolivia, La Paz, Bolivia

**Timeline**

- 2400 to 1900 BCE: Babylonians record equations on clay tablets
- 1850 BCE: Egyptian Rhind papyrus collects mathematical problems
- 400 BCE: reckoning boards and tables used
- 330 to 31 BCE: Egyptian mathematical problem papyrus
- 150 BCE: Jain mathematicians write the Shikharang Sutra
- 320: Al-Khwarizmi's writings include Hindu-Arabic numerals
- 1469 to 1499: Handwritten manuscripts become bound books
- 1609: Early slide rule developed
- 1617: Napier's Rahnologie calculates with rods and rings
- 2000 to 1650 BCE: Egyptian scribes record mathematical problems on papyrus
- 1046 to 356 BCE: Chinese mathematical text Zhoubi Suanjing
- 370 to 256 BCE: Eudoxus of Rhodes composes History of Arithmetic
- 300 BCE: Seleucia tablet (oldest known surviving counting board)
- 300 to 1000: Bakhshali manuscript (oldest known use of zero symbol)
- 1440 to 1490: Printing press invented
- 1484: Petrus Ramus publishes Renaissance mathematics
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- 1619 to 1622: Babbage difference engine

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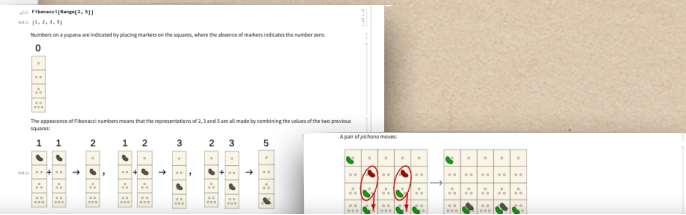
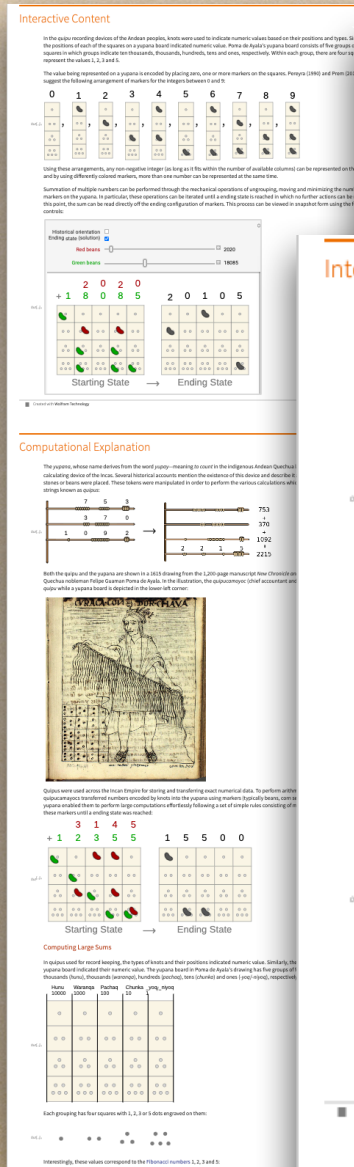
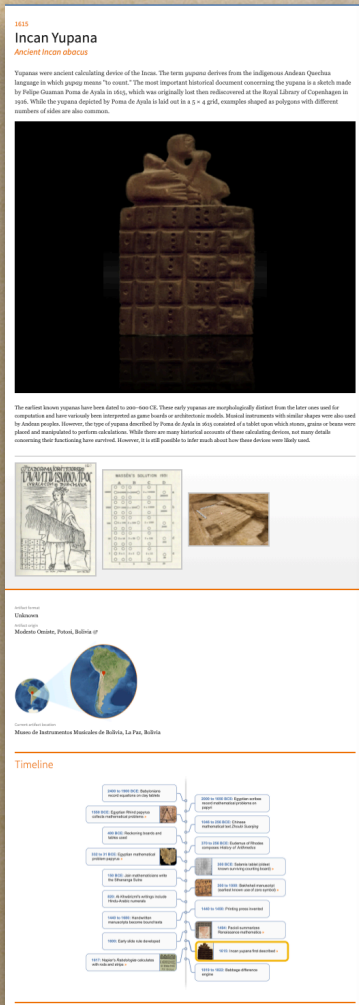


# Interactive and Computational Content

- ◆ Interactive content gives a Manipulate-based exploration of artifact content including some basic background and information
- ◆ Computational explanations give detailed explanations of the mathematical content of the artifact that make extensive use of the Wolfram Language



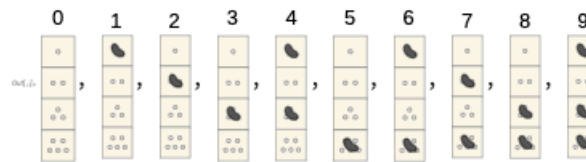
## Each Math Artifact Page Contains:



## Interactive Content

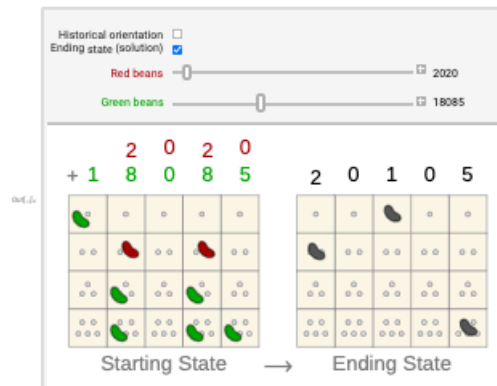
In the quipu recording devices of the Andean peoples, knots were used to indicate numeric values based on their positions and types. Similarly, the positions of each of the squares on a yupana board indicated numeric value. Poma de Ayala's yupana board consists of five groups of four squares in which groups indicate ten thousands, thousands, hundreds, tens and ones, respectively. Within each group, there are four squares that represent the values 1, 2, 3 and 5.

The value being represented on a yupana is encoded by placing zero, one or more markers on the squares. Pereyra (1990) and Prem (2019) suggest the following arrangement of markers for the integers between 0 and 9:

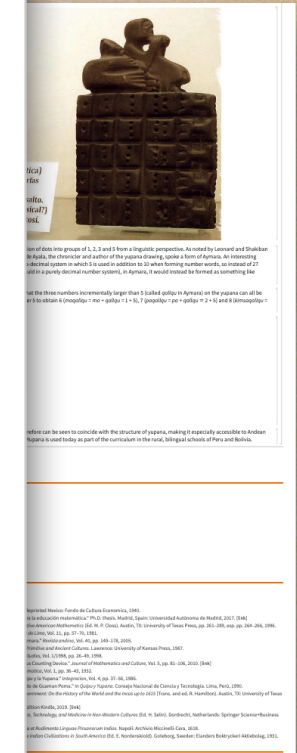


Using these arrangements, any non-negative integer (as long as it fits within the number of available columns) can be represented on the yupana, and by using differently colored markers, more than one number can be represented at the same time.

Summation of multiple numbers can be performed through the mechanical operations of ungrouping, moving and minimizing the number of markers on the yupana. In particular, these operations can be iterated until a ending state is reached in which no further actions can be made. At this point, the sum can be read directly off the ending configuration of markers. This process can be viewed in snapshot form using the following controls:



Created with Wondershare Technology

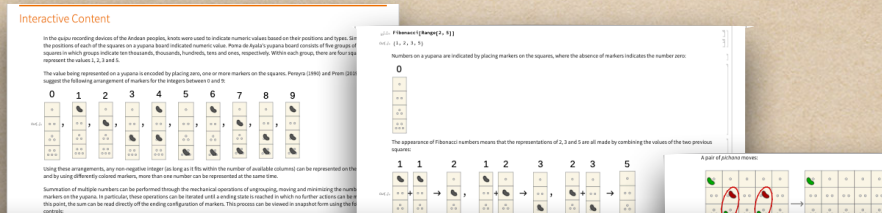
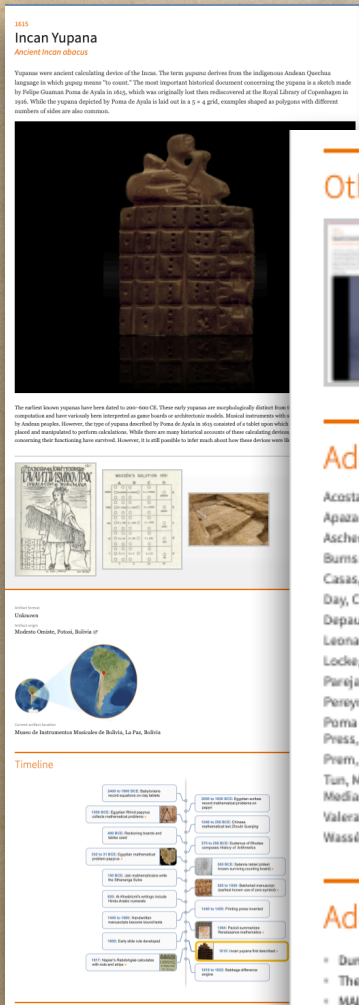








## Each Math Artifact Page Contains:



## Other Resources



### Additional Reading

Acosta, J. *La Historia Natural y Moral de Los Indias*. 1596. Reprinted Mexico: Fondo de Cultura Económica, 1940.

Apaza Luque, H. J. "La yupana, material manipulativo para la educación matemática." Ph.D. thesis. Madrid, Spain: Universidad Autónoma de Madrid, 2017. [link]

Ascher, M. "Mathematical Ideas of the Incas." Ch. 10 in *Native American Mathematics* [Ed. M. P. Cross]. Austin, TX: University of Texas Press, pp. 261–289, esp. pp. 264–266, 1996.

Burns Glyn, W. "La tabla de cálculo de los Incas." *Boletín de Lima*, Vol. 11, pp. 57–70, 1981.

Casas, G. P. "Los Sistemas Numericos Del Quechua y el Aymara." *Revista andina*, Vol. 40, pp. 149–178, 2005.

Day, C. *Quipus and Witches' Knots: The Role of the Knot in Primitive and Ancient Cultures*. Lawrence: University of Kansas Press, 1967.

Depaulis, T. "Inca Dice and Board Games." *Board Games Studies*, Vol. 1/1998, pp. 26–49, 1998.

Leonard, M. and Shalikian, B. "The Incan Abacus: A Curious Counting Device." *Journal of Mathematics and Culture*, Vol. 5, pp. 81–106, 2010. [link]

Locke, L. L. "The Ancient Peruvian Abacus." *Scripta Mathematica*, Vol. 1, pp. 36–43, 1932.

Pereja, D. "Instrumentos Prehispánicos de Cálculo: el Quipu y la Yupana." *Integración*, Vol. 4, pp. 37–56, 1986.

Pereyra, H. "El Antiguo abaco peruano según el manuscrito de Guamán Poma." in *Quipu y Yupana*. Consejo Nacional de Ciencia y Tecnología. Lima, Perú, 1990.

Poma de Ayala, G. F. *The First New Chronicle and Good Government: On the History of the World and the Incas up to 1615* (Trans. and ed. R. Hamilton). Austin, TX: University of Texas Press, 2009.

Prem, D. *Yupana Inka: Decoding the Inka's Math*. English Edition Kindle, 2019. [link]

Tun, M. "Yupana." in *Encyclopedia of the History of Science, Technology, and Medicine in Non-Western Cultures* [Ed. H. Selin]. Dordrecht, Netherlands: Springer Science+Business Media, 2014. [link]

Valera, B. *Exault Imperialis Blas Valera Populo Suo e Historia et Rudimento Linguae Piraurunorum Indios*. Napoli: Archivio Miccinelli-Cera, 1618.

Wassén, H. "The Ancient Peruvian Abacus." In *Origin of the Indian Civilizations in South America* [Ed. E. Nordenskiöld]. Göteborg, Sweden: Elanders Boktryckeri Aktiebolag, 1931.

### Additional Links

- Dumbarton Oaks Reseach Library and Collections: The Role of Knots
- The Guaman Poma Website: A Digital Research Center of the Royal Library, Copenhagen, Denmark
- MALI, Museo de Arte de Lima: Yupana-Inca style 1400 AD - 1532 AD
- Metropolitan Museum of Art: Game Board (yupana)
- Museo Chileno de Arte Precolombino: Yupas and pallares: Stones to calculate
- Wikipedia: Yupana
- YouTube: Yupana Inka - Competencia Método Tawa Pukllay vs Método Anábigo





# 8 Learning Journeys

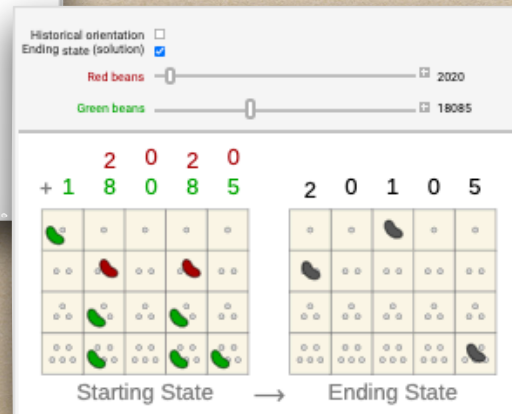
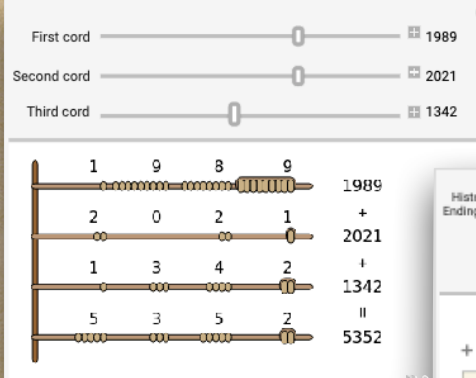
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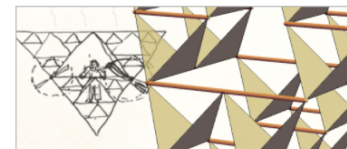
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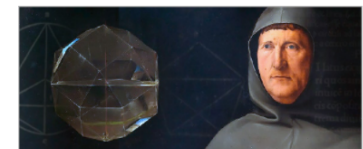
Show Your Work!



Squaring the Apsamikku Circle



Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles



Ancient Games of Chance

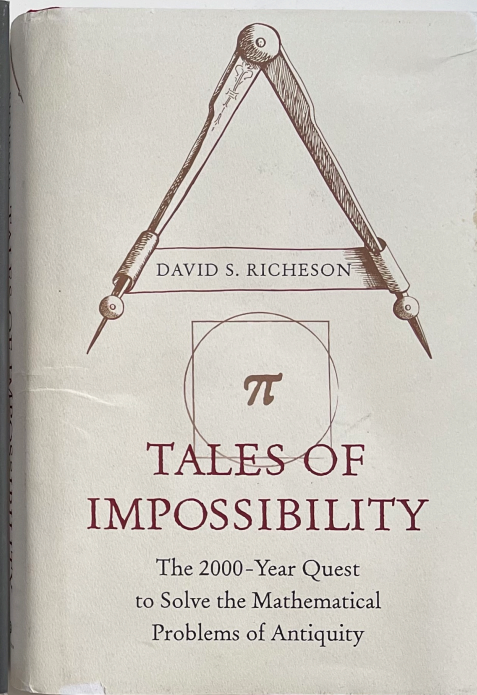
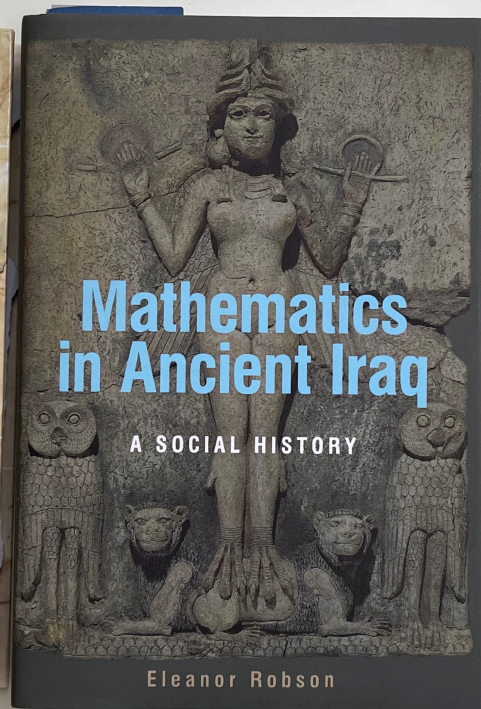
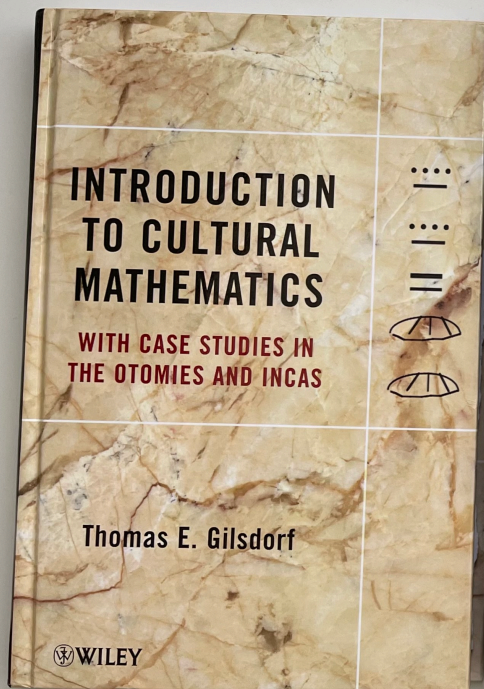


# Learning Journeys

- ◆ Intended as an engaging and fun "journey" through mathematical artifacts
- ◆ Aimed at students and other virtual museum visitors who are interested in the "mathematical story"
- ◆ Useful for classroom exploration or as a teaching tool
- ◆ Contain images and links to individual artifacts
- ◆ Include interactive content
- ◆ Primarily visual and descriptive with minimal mathematics
- ◆ 8 learning journeys



# Learning Journeys





# 8 Learning Journeys

- ◆ Mathematical Beans and Knotted Strings
- ◆ Balancing Ducks, Frogs and Grasshoppers
- ◆ Show Your Work!
- ◆ Squaring the Apsamikku Circle
- ◆ Making Machines Fly
- ◆ The Mathematics of a Masterpiece
- ◆ Ancient Right Triangles
- ◆ Ancient Games of Chance



Mathematical Beans and Knotted Strings



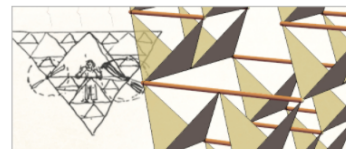
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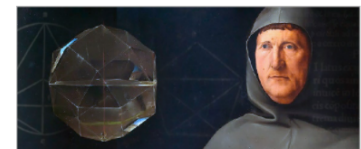
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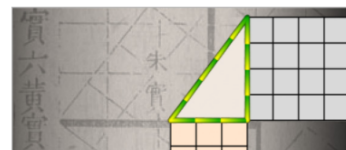
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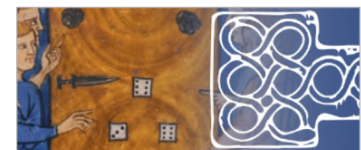
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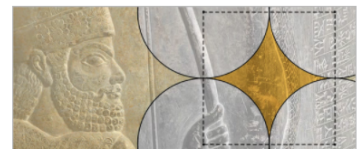
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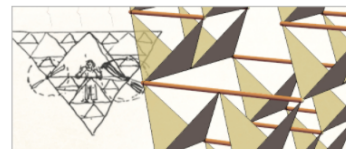
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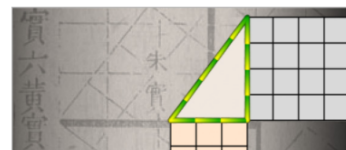
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# Ancient Games of Chance



# Ancient Games of Chance

Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")

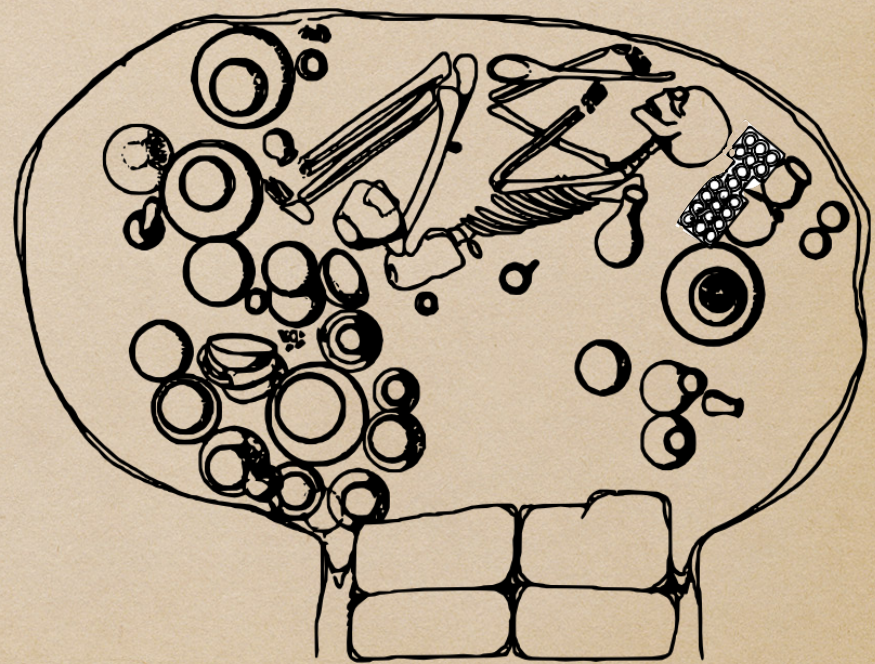
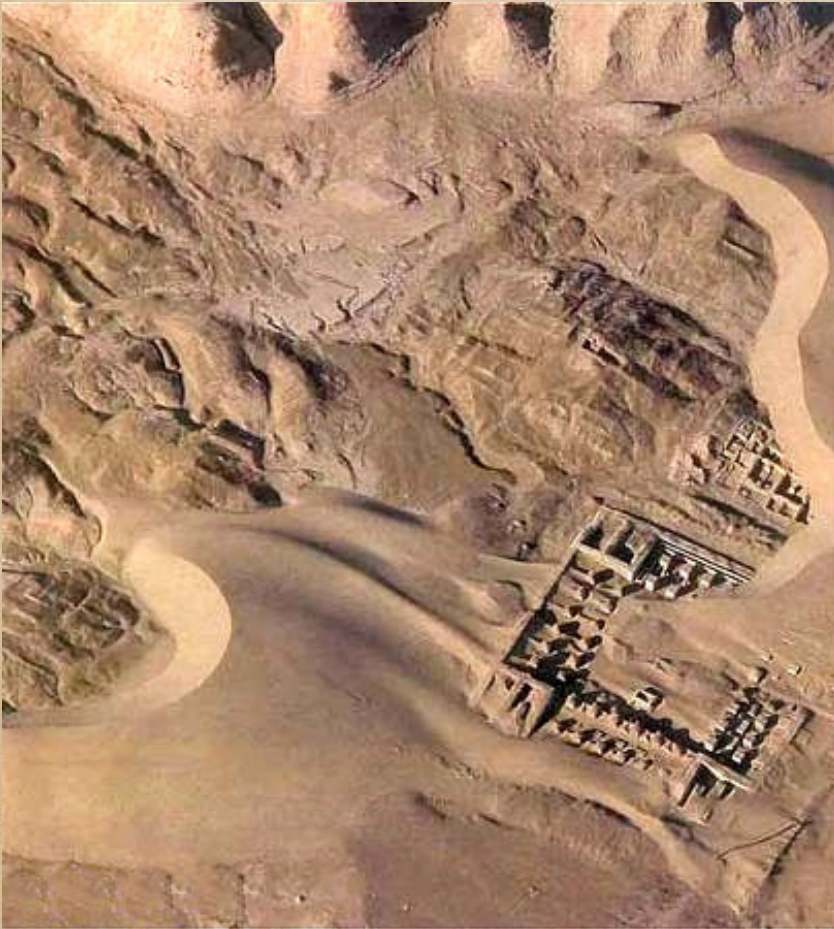


Dice crafted about 4500 years ago and discovered in the 1970s by an Italian expedition to the Burnt City ruins located in nowadays Iran, midway between the Middle East and the Indus Valley, India.



# Ancient Games of Chance

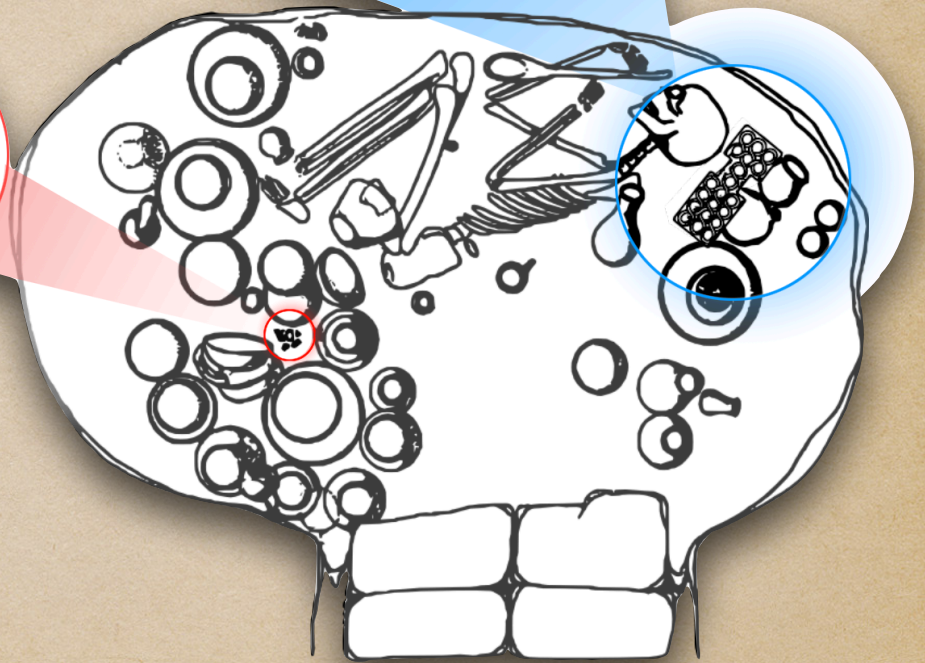
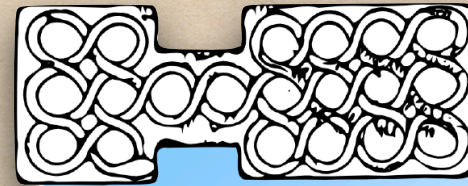
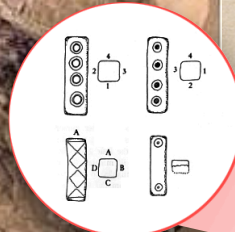
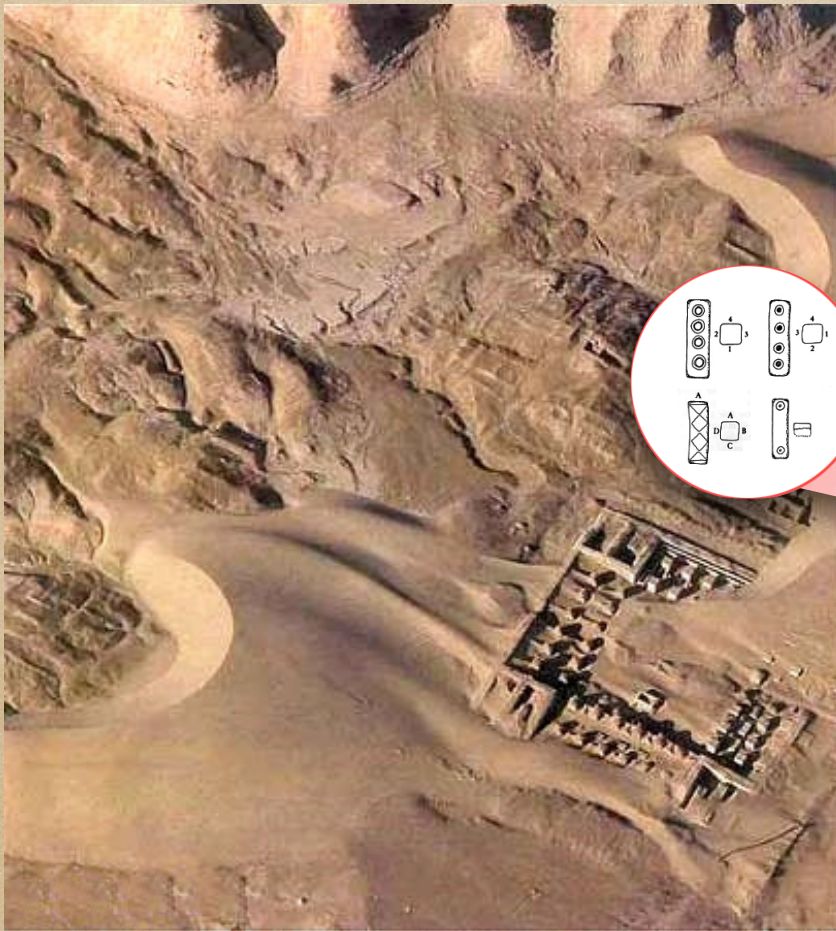
Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")





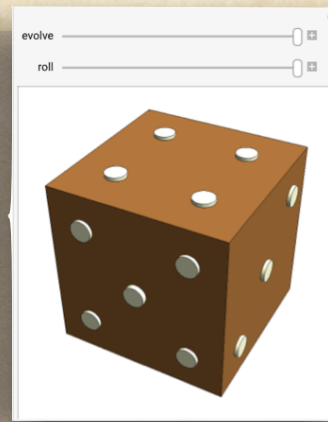
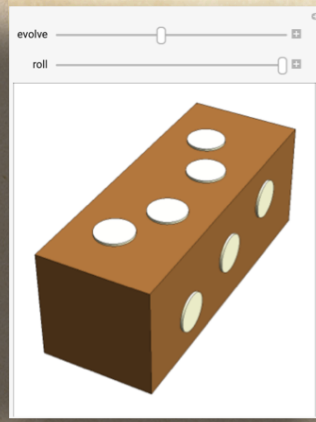
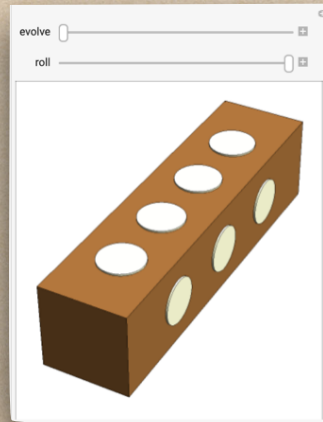
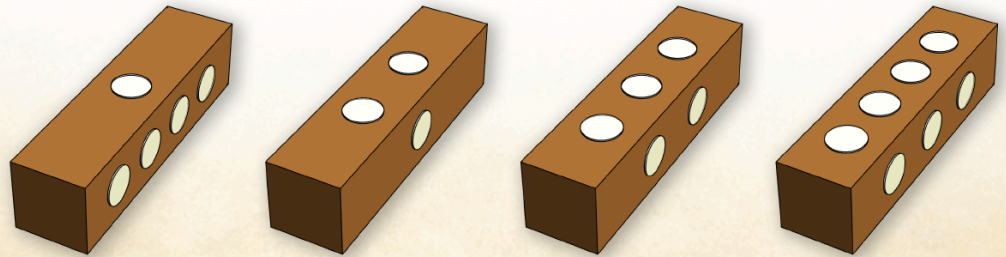
# Ancient Games of Chance

Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")





# Ancient Games of Chance



Four-valued die from the Burnt City  
morphed to a modern six-sided cubical die.



# Ancient Games of Chance

Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")

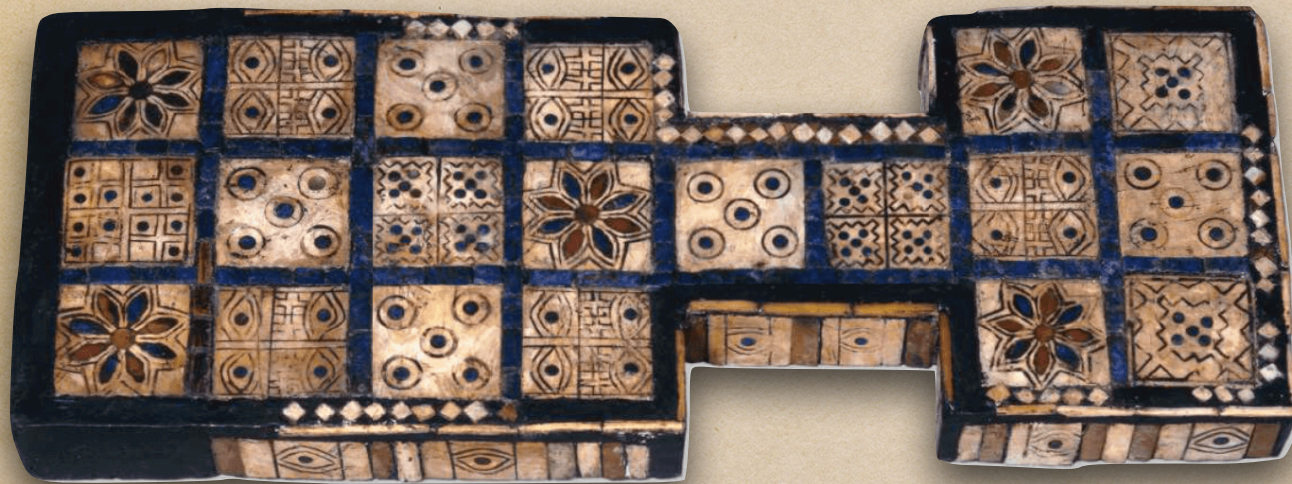


The game board was adorned with a knotted snake carved in relief and is identical in layout to a board found in the Royal Cemetery at Ur.



# Ancient Games of Chance

Royal Cemetery at Ur (British Museum), 2600 BCE



Game board from the Royal Cemetery at Ur (British Museum item #120840), which has been dated to 2600 BCE. While the Ur board is identical in form to the Burnt City board, it is much more regally adorned, including many beautiful and intricate geometric patterns.



# Ancient Games of Chance

Royal Cemetery at Ur (British Museum), 2600 BCE



While the Game of Twenty Squares is the world's oldest known board game, it can still be played today, since its rules have been deciphered by Dr. Irving Finkel, assistant keeper of ancient Mesopotamian script, languages and cultures in the Middle East department at the British Museum.



# Ancient Games of Chance

Egyptian game box, ca. 1635–1458 BCE.



Sheep knucklebones were used as a randomizing device because it has four long sides on which it can land when cast, with the numerical value assigned to the side facing up.



# Ancient Games of Chance

Egyptian game box, ca. 1635–1458 BCE.



Sheep knucklebones were used as a randomizing device because it has four long sides on which it can land when cast, with the numerical value assigned to the side facing up.

White stone die. 30 BCE–364 CE. Roman period.



# Ancient Games of Chance

The process of making fair dice from Folio 65v of the hand-illuminated manuscript the *Book of Games*, or *Libro de axedrez, dados e tablas* (Book of Chess, Dice and Tables, in Old Spanish).



During the Greco-Roman period, cubic dice became more common and gradually replaced throwing sticks and knucklebones for use with board games.



# Ancient Games of Chance













Two winning triga rolls. Libro de axedrez, dados e tablas, Fol. 66r.

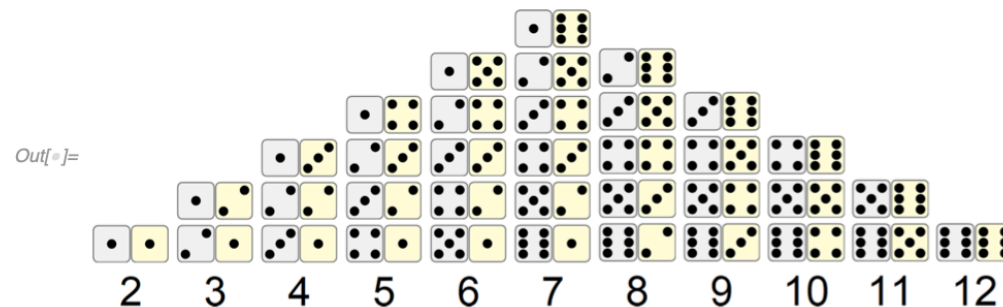




# Ancient Games of Chance

The theory of probability was born when the nobles, in the 17th century, commissioned the scientists of the time to solve the various questions that arise in games of chance, in particular the game of dice. Galileo Galilei's paper "About the Discoveries of Dice" dates from 1596, in which he investigates how by throwing three dice, some scores are more advantageous than others.

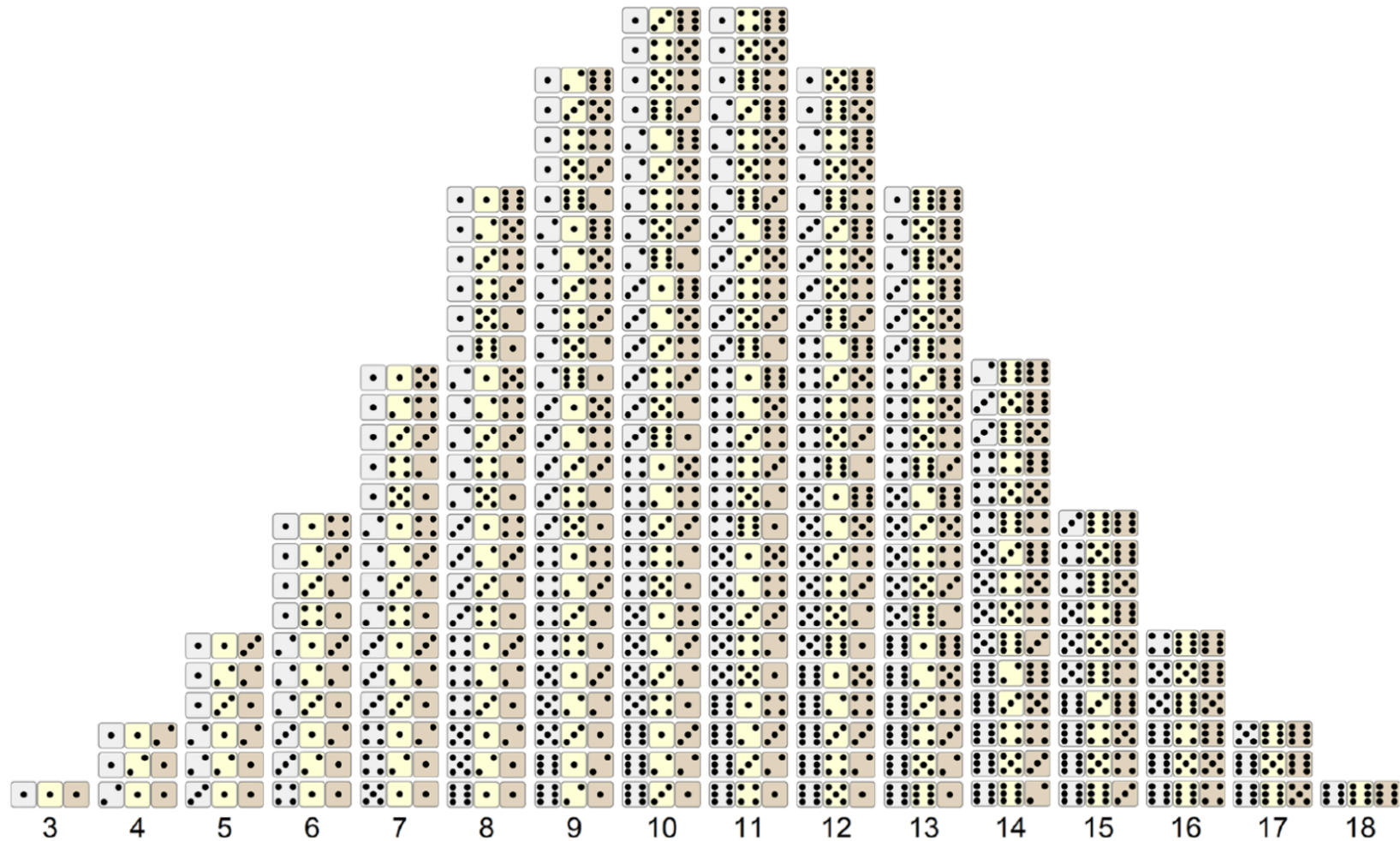
```
probabilities=GroupBy[Tuples[{
{1→,2→,3→,4→,5→,6→},
{1→,2→,3→,4→,5→,6→}],#[[1,1]]+#[[2,1]]&];
GraphicsRow[Rasterize@Labeled[Column[Row[{#[[1,2]],#[[2,2]]]&/@#[[2]],Spacings→0.3],
Style[Text[ToString[#[[1]]],32]]&/@Normal[probabilities],
Alignment→Bottom]
```



*The probabilities for obtaining a given total using two dice.*



# Ancient Games of Chance



The probabilities for obtaining a given total using three dice, which approaches a *normal distribution*.



# Ancient Games of Chance

Galileo stated that with three dice, there can only be one way of obtaining a 3 (1, 1, 1) and an 18 (6, 6, 6). However, there are three combinations for obtaining a 6—(4, 2, 1), (3, 2, 1) and (2, 2, 2)—which can occur in different orders, making 10 possibilities. There are four combinations for a 7—(5, 1, 1), (4, 2, 1), (3, 3, 1) and (3, 2, 2)—which lead to 15 possibilities. However, although 9 and 12 could be made up in the same number of ways as 10 and 11, from their experience, gamblers claimed that the occurrence of 10 and 11 were more likely! Galileo showed that the total number of possible throws with three dice are 216, and he gave a table of the number of possible throws for a total of 10, 9, 8, 7, 6, 5, 4 and 3, showing that the throws for 11 to 18 were symmetrical with these. In this way, he showed that there were 27 possible throws to obtain a 10, and 25 for a 9.



# Ancient Games of Chance

## Ancient Games of Chance

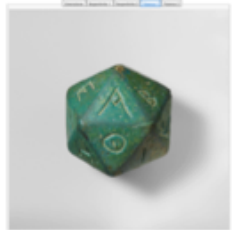
Spreads of chance games have fascinated humanity, stretching back over 6,000 years. From the earliest civilizations to the modern world, people have played games of chance for entertainment, social interaction, and even as a means of divination.

Some of the most famous ancient games of chance include the game of Ur, played with a board and dice, and the game of Senet, played with a board and dice. These games were often played by the elite, and their rules and strategies were passed down through generations.


The game of Ur, for example, was played on a board with 14 squares. The player moved a piece from one square to another by rolling a die. The game was often played by the king and his advisors, and it was considered a test of skill and strategy.

The game of Senet, on the other hand, was played on a board with 30 squares. The player moved a piece from one square to another by rolling a die. The game was often played by the king and his advisors, and it was considered a test of skill and strategy.


These ancient games of chance have inspired modern games of chance, such as roulette, craps, and slot machines. They have also inspired the development of probability theory and statistics, which are essential tools for understanding the outcomes of chance events.




A green 12-sided die with numbers 1 through 12. The die is shown in a 3D perspective view, highlighting its geometric shape and the arrangement of the numbers.




A collection of ancient dice, including a 12-sided die and several 6-sided dice. The dice are shown in a 3D perspective view, highlighting their various shapes and sizes.



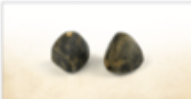
A diagram showing the layout of a game board with various squares and pieces. The diagram illustrates the movement of pieces across the board, with arrows indicating the direction of play.



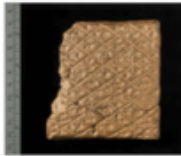
A collection of ancient game boards, including a board with a grid of squares and a board with a grid of circles. The boards are shown in a 3D perspective view, highlighting their intricate designs and the arrangement of the squares or circles.




A diagram showing the layout of a game board with various squares and pieces. The diagram illustrates the movement of pieces across the board, with arrows indicating the direction of play.




A collection of ancient game boards, including a board with a grid of squares and a board with a grid of circles. The boards are shown in a 3D perspective view, highlighting their intricate designs and the arrangement of the squares or circles.




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# 8 Learning Journeys

- ◆ Mathematical Beans and Knotted Strings
- ◆ Balancing Ducks, Frogs and Grasshoppers
- ◆ Show Your Work!
- ◆ Squaring the Apsamikku Circle
- ◆ Making Machines Fly
- ◆ The Mathematics of a Masterpiece
- ◆ Ancient Right Triangles
- ◆ Ancient Games of Chance



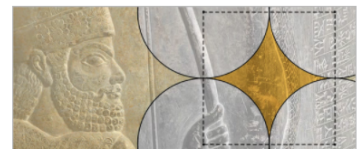
Mathematical Beans and Knotted Strings



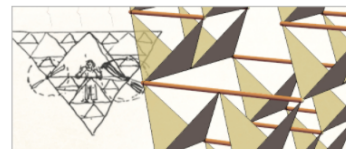
Balancing Ducks, Frogs and Grasshoppers



Show Your Work!



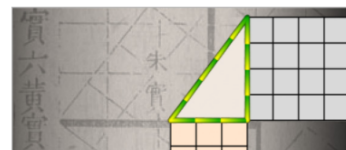
Squaring the Apsamikku Circle



Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles

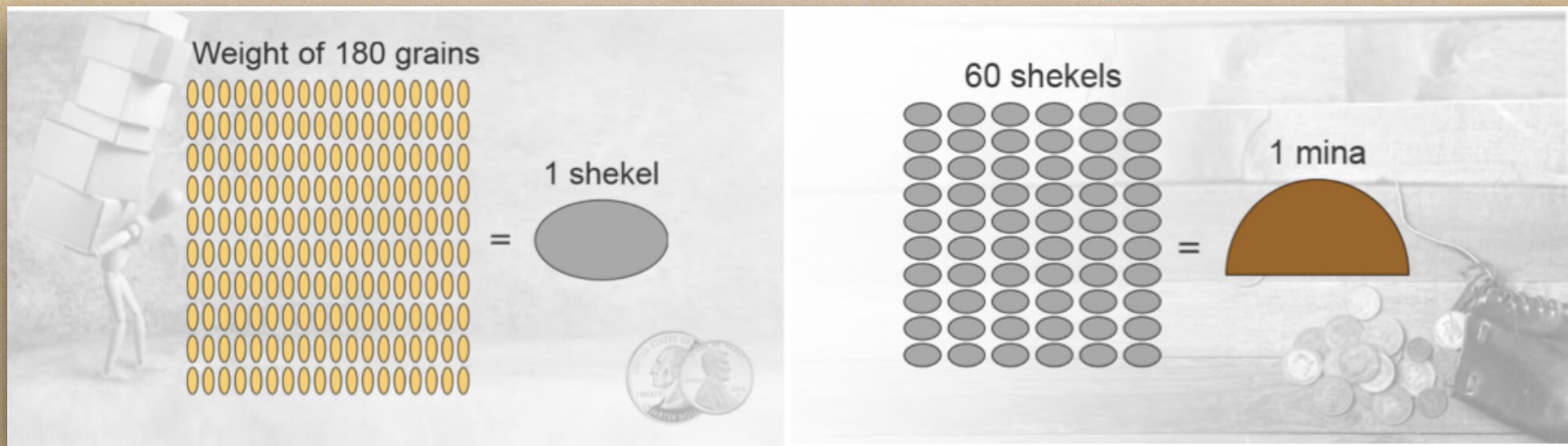


Ancient Games of Chance



# Balancing Ducks, Frogs and Grasshoppers

Barley was so important to the ancient Mesopotamians that a barley grain was used as the smallest unit of length, area, volume and weight. A shekel of silver weighed as much as 180 barley grains, or about 8.4 grams. 60 shekels weighed 1 mina, and 60 mina weighed 1 talent.





# Balancing Ducks, Frogs and Grasshoppers

Merchants would carry around their own set of weights to help them with trading. Most weights were sort of grain-shaped. Mesopotamian weights were often made of polished **hematite**:



*Hematite weights ranging from three shekels to one mina. Uruk, Mesopotamia, ca. 2000–1600 BCE.*

Mesopotamian weights were often shaped like a sleeping duck, with its neck and head resting on its back:



*Left: Duck-shaped hematite weights, Mesopotamia, ca. 2000 BCE.*

*Right: A sleeping duck!*

There are rare examples of Mesopotamian weights in other shapes, but most weights were either grain-shaped or duck-shaped. Here are some unusual examples of Mesopotamian weights: a grasshopper, a shell and a cute frog:



*Left: Mesopotamian grasshopper weight made of hematite, ca. 1800–1600 BCE.*

*Center: Mesopotamian shell weight made of hematite, ca. 1800–1600 BCE.*

*Right: Mesopotamian frog weight, ca. 2000–1600 BCE. The Akkadian inscription under the frog's throat reads: "a frog [weighing] 10 mina, a legitimate weight of the god Shamash, belonging to Iddin-Nergal, son of Arkat-ili-damqa."*



# Balancing Ducks, Frogs and Grasshoppers

- ◆ To understand how these weight stones might have been used I created the following balance scale interactive. As you add more barley grains on the left side of the scale, the merchant adds duck weights that come in fractions of a shekel so the sides balance. The beam at the top of the scale acts as an "equal" sign!



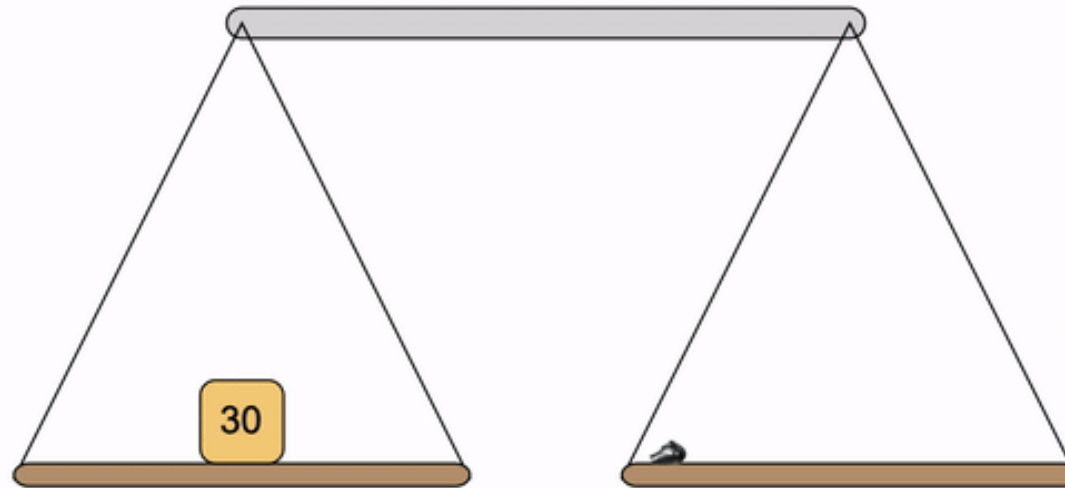
Bala

Weight units: **Grams** Ounces

Barely grains: 

1.4 g

$$\frac{1}{6} = \frac{1}{6}$$



$$\frac{30}{180} = \frac{1}{6} \text{ shekel}$$

$$\frac{1}{6}$$

top of the scale acts as an "equal" sign!



# Balancing Ducks, Frogs and Grasshoppers

PROBABILITY & WOLFRAM

HISTORY OF  
MATHEMATICS  
PROJECT

15 | Learning Journey 1

8 | Measurement 1

around  
3000–1200 BCE

## Balancing Ducks, Frogs and Grasshoppers

Weights and Measures in Ancient Mesopotamia

If you were a worker thousands of years ago, how would you get paid?

You wouldn't be paid in coins, because coins were not invented until around 600 BCE.

You wouldn't be paid in paper money, either, since paper money was not developed until around 960 CE, in China.

In ancient Mesopotamia (present-day Iraq) over 5000 years ago (around 3000–1200 BCE), people measured a grain called barley as payments for their work.

Barley is a grain like wheat. You can cook barley and eat it, like you would eat a bowl of oatmeal, or you can grind it into flour and use the flour to make bread.

A stalk of barley. Harvested barley grain.

Four thousand years ago, a worker in Mesopotamia would have been paid *one gur* of grain every month. That's about 380 liters of grain (or what's equivalent to 150 cups) of grain. So for one day of work, a person could earn about 3.2 liters of grain (or 13 cups) of grain every 30 days. You probably can't even eat that much grain in one day. You could use it to feed your family, or you could get a sack filled with the new little bottles of glass you saw in your payment.

If you got one liter of barley grain, you get a little more than a cup of barley flour (flour is fluffier than grain). That's enough flour to make two loaves of bread with a 10-centimeter (4-inch) loaf. So if a worker could earn 3.2 liters of grain in a day, they would potentially make 20 loaves of bread with the grain they earned in one day! Even if you had a big family, you probably wouldn't want to make 20 loaves of bread every single day!

People would use some of the grain to feed their families. They would use the rest for trade, or they would use it to make in ancient Mesopotamia, grain was like money. People could trade grain for tools like beams, steels, vegetable oil, dates and onions, or they could buy supplies. Since grain takes a long time, they could use it for food storage. They could also use the grain they saved to trade for things like, just like we save our money today.

The Mesopotamians traded barley to buy everything they locally. But large quantities of barley are heavy; the weight of one gur is more than the heaviest pound if people wanted to use it for anything expensive. It wasn't practical to store or transport thousands of pounds worth of barley. So they used silver to trade for more valuable things like sheep, houses or land. Sometimes they used a combination of silver and barley to make payments.

Barley was so important to the ancient Mesopotamians that a barley grain was used as the smallest unit of length, area, volume and weight. One gur of grain could be exchanged for one *shekel* (present-day 300–450 g) of silver. The word *shekel* means "to weigh out." A shekel of silver weighed as much as 360 barley grains, or about 4 g of grain. That's only a little more than the weight of one quarter coin (see page 9).

A shekel was a lump of silver that just weighed 360 grams—that doesn't mean that you could trade 360 actual grams of barley for a shekel! One hundred eighty grams is a handful. Remember, you would need to trade one gur (or the hundred pounds) of barley for a shekel. It would take three days to carry one gur of grain. A shekel of silver was definitely more valuable than one gur of grain.

60 shekels weighed one *mina*, or about 2 pounds.

The word *mina* means "to count." It would take an ordinary worker at least five months to earn that much silver!

60 minas weighed one *talent*.

One talent weighed about 67 pounds. A talent was a huge amount of silver. In today's terms, a talent of silver would be worth about \$12,000.

Merchants would carry around their own set of weights to help them with trading. Most weights were out of grain-shaped. Weasaponnet weights were often made of polished hematite.



weasaponnet weights ranging from three shawls to one mina. (c. 1000 BCE, Mesopotamia, ca. 2000-2000 BCE)

Weasaponnet weights were often shaped like a sleeping duck, with its neck and head resting on its back:



Left: Duck-shaped weasaponnet weights. Mesopotamia, ca. 2000 BCE.  
Right: A weasaponnet.

There are rare examples of Mesopotamian weights that are shaped like birds, but most weights were either grain-shaped or duck-shaped. Here are some unusual exceptions to Mesopotamian weights: a guinea pig, a bull's head and a scale frog.



Left: Mesopotamian guinea pig weight made of hematite, ca. 2000-2000 BCE.  
Middle: Mesopotamian bull's head weight, ca. 2000-2000 BCE.  
Right: Mesopotamian scale frog, ca. 2000-2000 BCE. The scale frog was used to measure under the frog's throat ready. It was hanging through 10 minis, an appropriate weight of 10 minis, because hanging by a scale of 10 minis was the same as 10 minis.

Merchants would use a simple balance scale to see how much something weighed. They would put the item to be traded on one side of the scale, and add or subtract weights until the scale balanced:



Antique balance scale.

Other ancient civilizations also used balance scales and weights in the shape of animals.



Left: Copper, an Egyptian bronze balance scale, ca. 1000 BCE. The weight on the right is a small statue of a bird, which weighs an amount that is about half the weight of the scale. The weight on the left is a small statue of a bird, which weighs an amount that is about half the weight of the scale. The weight on the right is a small statue of a bird, which weighs an amount that is about half the weight of the scale. The weight on the left is a small statue of a bird, which weighs an amount that is about half the weight of the scale.

The Mesopotamians grouped things by the number 60 (60 shawls = 1 mina, 60 minis = 1 talent, etc.). This number system is called base 60, or sexagesimal. The number 60 comes from the numbers 2, 3, 4, 5, 6, 10, 12, 15, 30 and 60. The divisibility of 60 makes it a perfect base for doing arithmetic, were there some different ways you can arrange 60 objects to include:



Here's one way you can count it on your finger! Each finger has three bones. Using your right hand, you can count 60 things like this:



© 2014 by Math Learning Technology

[illegible]





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# Benefits of a Wolfram Language build system

- ◆ Easy to curate all relevant data (text, images, metadata, mathematical and interactive content) in a single place: notebooks
- ◆ Content elements such as maps, timelines, and thumbnails can be generated completely programmatically using built-in Wolfram Language functionality
- ◆ Incremental builds are easy simply by checking for changed notebook content



# Thanks!

- ◆ Overdeck Family Foundation
- ◆ MoMath, the Museum of Mathematics in New York City.
- ◆ Stephen Wolfram
- ◆ Andrea Gerlach, Eric Weisstein, Sarah Keim Williams
- ◆ Lori Goodman, Sushma Kini (project management)
- ◆ Michael Trott (content suggestions and review), Christopher Wolfram (content suggestions and discussions), Dan McDonald (synthetic geometry contributions), MinHsuan Peng (custom timelines), Shadi Ashnai and Giulio Alessandrini (image processing)
- ◆ Heidi Kellner and Jeremy Davis (web design), Marion Morris (web implementation), Taylor Birch (proofreading)
- ◆ Our network of 50+ domain and content experts



# Background and Timeline

- ◆ In 2019, Stephen Wolfram proposed a project to develop a virtual interactive collection of mathematical artifacts for the Museum of Mathematics (MoMath) in New York City
- ◆ The project was generously funded by Overdeck Family Foundation
- ◆ Over the last two years, researchers at Wolfram Research have investigated and written up detailed histories, descriptions, and explanations for a collection of mathematical artifacts
- ◆ The results have been incorporated into a website ([history-of-mathematics.org](http://history-of-mathematics.org)) created using a custom build system modeled after the one being used for Stephen Wolfram's Physics Project



# Build system

- ◆ Website is built using the Wolfram Language
- ◆ Source documents are tagged notebooks [example]
- ◆ All content built to and hosted in the Wolfram Cloud
- ◆ Computational/interactive content are simply notebook sections embedded directly in the cloud using WolframNotebookEmbedder
- ◆ Core workflow based on XMLTemplate + ExportForm:



# Build system

- ◆ Website is built using the Wolfram Language
- ◆ Source documents are tagged notebooks [example]
- ◆

```
CloudDeploy[
  ExportForm[
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    {"Text", "HTML"}
  ],
  $MathArtifactsURLBase <> url,
  Permissions → "Public"]
```
- ◆ Core workflow based on XMLTemplate + ExportForm:



# 8 Learning Journeys

- ♦ Mathematical Beans and Knotted Strings. Counting Methods from the Moche Culture.
- ♦ Balancing Ducks, Frogs and Grasshoppers. Weights and Measures in Ancient Mesopotamia.
- ♦ Show Your Work! Doing Math Homework on Clay Tablets, Papyri, Wax Tablets, Bamboo Strips and Birch Bark.
- ♦ Squaring the Apsamikku Circle. The Search to Solve One of the Oldest Problems in Math.
- ♦ Making Machines Fly. Overcoming the Square-Cube Law.
- ♦ The Mathematics of a Masterpiece. Portrait of Luca Pacioli.
- ♦ Ancient Right Triangles. The Pythagorean Theorem and the Gou-Gu Rule.
- ♦ Ancient Games of Chance. The Beginnings of the Mathematical Theory of Probability.