EXCERPTED FROM

STEPHEN WOLFRAM
A NEW KIND OF SCIENCE

SECTION 4.3

Recursive Sequences

Recursive Sequences

In the previous section, we saw that it is possible to get behavior of considerable complexity just by applying a variety of operations based on simple arithmetic. In this section what I will show is that with the appropriate setup just addition and subtraction turn out to be in a sense the only operations that one needs.

The basic idea is to consider a sequence of numbers in which there is a definite rule for getting the next number in the sequence from previous ones. It is convenient to refer to the first number in each sequence as $f[1]$, the second as $f[2]$, and so on, so that the n^{th} number is denoted $f[n]$. And with this notation, what the rule does is to specify how $f[n]$ should be calculated from previous numbers in the sequence.

In the simplest cases, $f[n]$ depends only on the number immediately before it in the sequence, denoted $f[n-1]$. But it is also possible to set up rules in which $f[n]$ depends not only on $f[n-1]$, but also on $f[n-2]$, as well as on numbers still earlier in the sequence.

The table below gives results obtained with a few specific rules. In all the cases shown, these results are quite simple, consisting of sequences that increase uniformly or fluctuate in a purely repetitive way.

$f[n] = 1 + f[n-1],\ f[1] = 1$

(a)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	…

$f[n] = 1 - f[n-1],\ f[1] = 1$

(b)	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	…

$f[n] = 2 f[n-1],\ f[1] = 1$

(c)	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	131072	262144	524288	1048576	2097152	…

$f[n] = f[n-1] + f[n-2],\ f[1] = 1,\ f[2] = 1$

(d)	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765	10946	17711	28657	46368	75025	121393	…

$f[n] = f[n-1] - f[n-2],\ f[1] = 1,\ f[2] = 1$

(e)	1	1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	…

$f[n] = -f[n-1] + f[n-2],\ f[1] = 1,\ f[2] = 1$

(f)	1	1	0	1	-1	2	-3	5	-8	13	-21	34	-55	89	-144	233	-377	610	-987	1597	-2584	4181	-6765	10946	-17711	28657	-46368	…

Examples of some simple recursive sequences. The n^{th} element in each sequence is denoted $f[n]$, and the rule specifies how this element is determined from previous ones. With all the rules shown here, successive elements either increase smoothly or fluctuate in a purely repetitive way. Sequence (c) is the powers of two; (d) is the so-called Fibonacci sequence, related to powers of the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$. All rules of the kind shown here lead to sequences where $f[n]$ can be expressed in terms of a simple sum of powers of the form a^n.

But it turns out that with slightly more complicated rules it is possible to get much more complicated behavior. The key idea is to consider rules which look at numbers that are not just a fixed distance back in the sequence. And what this means is that instead of depending only on quantities like $f[n-1]$ and $f[n-2]$, the rule for $f[n]$ can also for example depend on a quantity like $f[n-f[n-1]]$.

There is some subtlety here because in the abstract nothing guarantees that $n-f[n-1]$ will necessarily be a positive number. And if it is not, then results obtained by applying the rule can involve meaningless quantities such as $f[0]$, $f[-1]$ and $f[-2]$.

$f[n] = 1 + f[n - f[n - 1]],\ f[1] = 1$

(a) 1 2 2 3 3 3 4 4 4 4 5 5 5 5 5 6 6 6 6 6 6 7 7 7 7 7 7 7 8 8 8 8 8 8 8 8 9 9 9 9 9 9 9 9 9 10 10 10 …

$f[n] = 2 + f[n - f[n - 1]],\ f[1] = 1,\ f[2] = 1$

(b) 1 1 3 3 3 5 3 5 5 5 7 5 7 5 7 7 7 9 7 9 7 9 7 9 9 9 11 9 11 9 11 9 11 9 11 11 11 13 11 13 11 13 11 13 …

$f[n] = f[f[n - 1]] + f[n - f[n - 1]],\ f[1] = 1,\ f[2] = 1$

(c) 1 1 2 2 3 4 4 4 5 6 7 7 8 8 8 8 9 10 11 12 12 13 14 14 15 15 15 16 16 16 16 16 17 18 19 20 21 21 22 23 …

$f[n] = f[n - f[n - 1]] + f[n - f[n - 2] - 1],\ f[1] = 1,\ f[2] = 1$

(d) 1 1 2 2 3 4 4 4 5 6 6 7 8 8 8 8 9 10 10 11 12 12 12 13 14 14 15 16 16 16 16 16 17 18 18 19 20 20 20 21 …

$f[n] = f[n - f[n - 1]] + f[n - f[n - 2]],\ f[1] = 1,\ f[2] = 1$

(e) 1 1 2 3 3 4 5 5 6 6 6 8 8 8 10 9 10 11 11 12 12 12 12 16 14 14 16 16 16 16 20 17 17 20 21 19 20 22 21 22 …

$f[n] = f[n - f[n - 1] - 1] + f[n - f[n - 2] - 1],\ f[1] = 1,\ f[2] = 1$

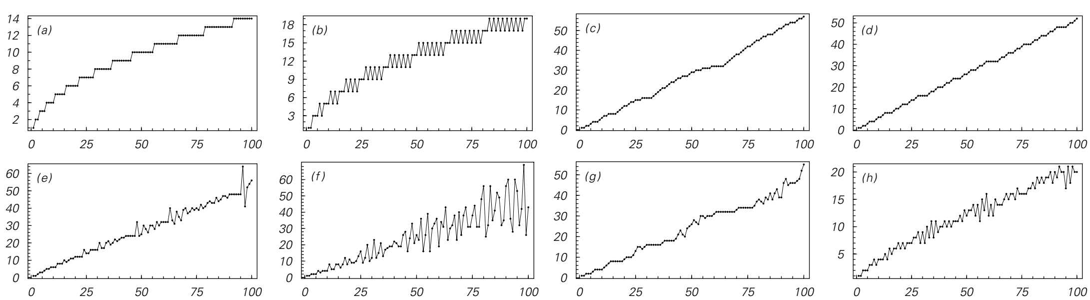
(f) 1 1 2 2 2 4 3 4 4 4 8 5 5 8 8 6 8 12 8 11 9 9 10 13 16 9 12 20 10 12 23 12 15 21 13 17 18 19 19 22 21 19 …

$f[n] = f[f[n - 1]] + f[n - f[n - 2] - 1],\ f[1] = 1,\ f[2] = 1$

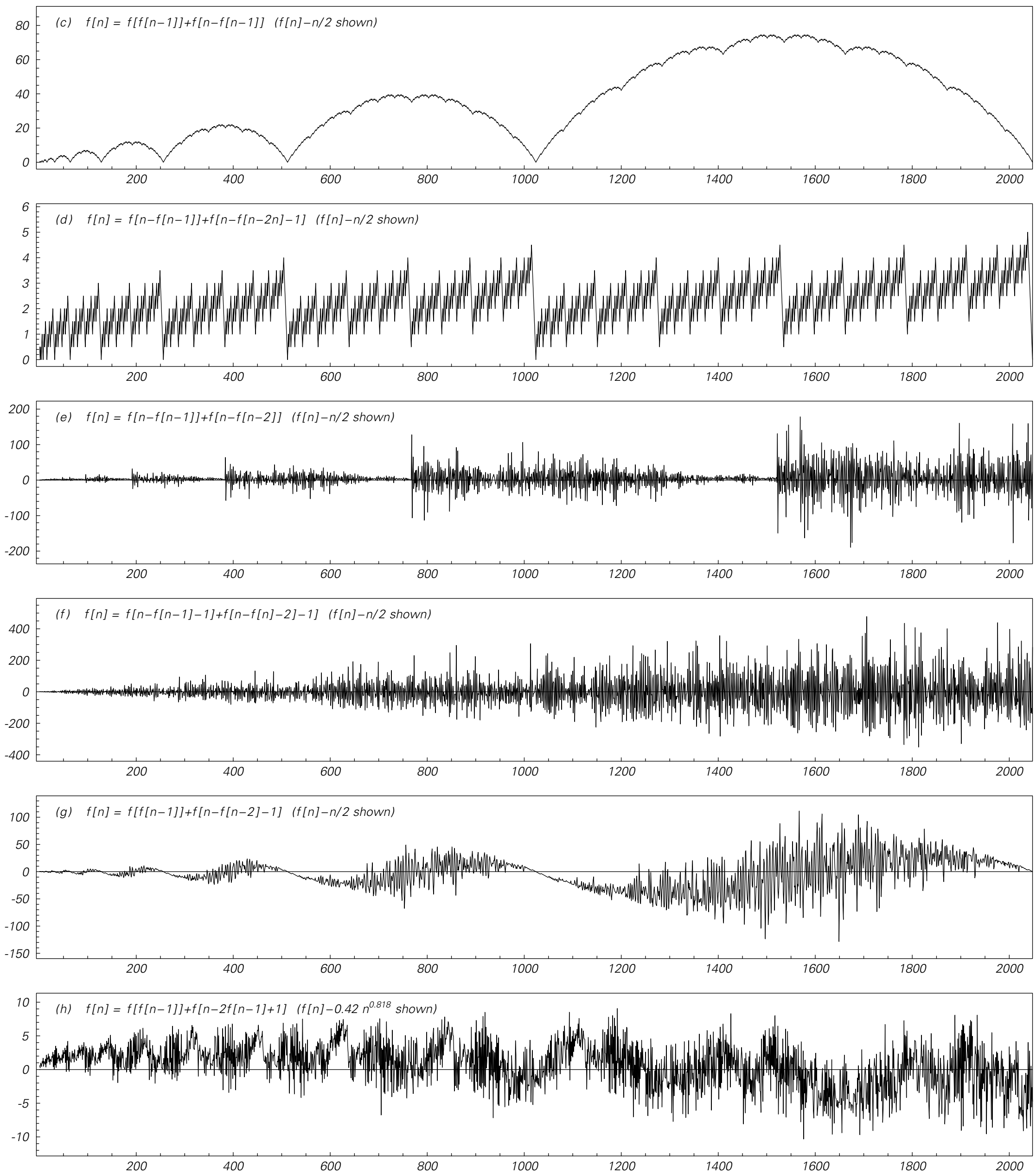
(g) 1 1 2 2 2 3 4 4 4 4 5 6 7 8 8 8 8 8 8 9 10 10 10 11 13 15 15 14 15 16 16 16 16 16 16 16 17 18 18 18 18 …

$f[n] = f[f[n - 1]] + f[n - 2 f[n - 1] + 1],\ f[1] = 1,\ f[2] = 1$

(h) 1 1 2 2 2 3 3 4 3 4 4 4 5 4 6 5 6 6 7 6 7 6 7 7 7 8 8 9 7 9 7 10 8 11 8 11 9 10 10 11 10 11 10 11 11 …



Examples of sequences generated by rules that do not depend only on elements a fixed distance back. Most such rules eventually end up involving meaningless quantities such as *f[0]* and *f[-1]*, but the particular rules shown here all avoid this problem.



Fluctuations in the overall increase of sequences from the previous page. In cases (c) and (d), the fluctuations have a regular nested form, and turn out to be directly related to the base 2 digit sequence of n. In the other cases, the fluctuations are more complicated, and seem in many respects random. All the rules shown start with $f[1] = f[2] = 1$.

For the vast majority of rules written down at random, such problems do indeed occur. But it is possible to find rules in which they do not, and the pictures on the previous two pages show a few examples I have found of such rules. In cases (a) and (b), the behavior is fairly simple. But in the other cases, it is considerably more complicated.

There is a steady overall increase, but superimposed on this increase are fluctuations, as shown in the pictures on the facing page.

In cases (c) and (d), these fluctuations turn out to have a very regular nested form. But in the other cases, the fluctuations seem instead in many respects random. Thus in case (f), for example, the number of positive and negative fluctuations appears on average to be equal even after a million steps.

But in a sense one of the most surprising features of the facing page is that the fluctuations it shows are so violent. One might have thought that in going say from $f[2000]$ to $f[2001]$ there would only ever be a small change. After all, between $n = 2000$ and 2001 there is only a 0.05% change in the size of n.

But much as we saw in the previous section it turns out that it is not so much the size of n that seems to matter as various aspects of its representation. And indeed, in cases (c) and (d), for example, it so happens that there is a direct relationship between the fluctuations in $f[n]$ and the base 2 digit sequence of n.

In case (d), the fluctuation in each $f[n]$ turns out to be essentially just the number of 1's that occur in the base 2 digit sequence for n. And in case (c), the fluctuations are determined by the total number of 1's that occur in the digit sequences of all numbers less than n.

There are no such simple relationships for the other rules shown on the facing page. But in general one suspects that all these rules can be thought of as being like simple computer programs that take some representation of n as their input.

And what we have discovered in this section is that even though the rules ultimately involve only addition and subtraction, they nevertheless correspond to programs that are capable of producing behavior of great complexity.