EXCERPTED FROM

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A NEW KIND OF SCIENCE

SECTION 4.9

Partial Differential Equations

Partial Differential Equations

By introducing continuous cellular automata with a continuous range of gray levels, we have successfully removed some of the discreteness that exists in ordinary cellular automata. But there is nevertheless much discreteness that remains: for a continuous cellular automaton is still made up of discrete cells that are updated in discrete time steps.

So can one in fact construct systems in which there is absolutely no such discreteness? The answer, it turns out, is that at least in principle one can, although to do so requires a somewhat higher level of mathematical abstraction than has so far been necessary in this book.

The basic idea is to imagine that a quantity such as gray level can be set up to vary continuously in space and time. And what this means is that instead of just having gray levels in discrete cells at discrete time steps, one supposes that there exists a definite gray level at absolutely every point in space and every moment in time—as if one took the limit of an infinite collection of cells and time steps, with each cell being an infinitesimal size, and each time step lasting an infinitesimal time.

But how does one give rules for the evolution of such a system? Having no explicit time steps to work with, one must instead just specify the rate at which the gray level changes with time at every point in space. And typically one gives this rate as a simple formula that depends on the gray level at each point in space, and on the rate at which that gray level changes with position.

Such rules are known in mathematics as partial differential equations, and in fact they have been widely studied for about two hundred years. Indeed, it turns out that almost all the traditional mathematical models that have been used in physics and other areas of science are ultimately based on partial differential equations. Thus, for example, Maxwell's equations for electromagnetism, Einstein's equations for gravity, Schrödinger's equation for quantum mechanics and the Hodgkin-Huxley equation for the electrochemistry of nerve cells are all examples of partial differential equations.

It is in a sense surprising that systems which involve such a high level of mathematical abstraction should have become so widely used in practice. For as we shall see later in this book, it is certainly not that nature fundamentally follows these abstractions.

And I suspect that in fact the current predominance of partial differential equations is in many respects a historical accident—and that had computer technology been developed earlier in the history of mathematics, the situation would probably now be very different.

But particularly before computers, the great attraction of partial differential equations was that at least in simple cases explicit mathematical formulas could be found for their behavior. And this meant that it was possible to work out, for example, the gray level at a particular point in space and time just by evaluating a single mathematical formula, without having in a sense to follow the complete evolution of the partial differential equation.

The pictures on the facing page show three common partial differential equations that have been studied over the years.

The first picture shows the diffusion equation, which can be viewed as a limiting case of the continuous cellular automaton on page 156. Its behavior is always very simple: any initial gray progressively diffuses away, so that in the end only uniform white is left.

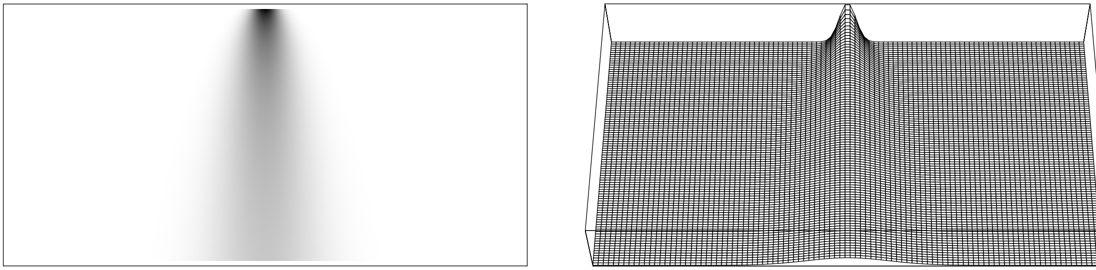
The second picture shows the wave equation. And with this equation, the initial lump of gray shown just breaks into two identical pieces which propagate to the left and right without change.

The third picture shows the sine-Gordon equation. This leads to slightly more complicated behavior than the other equations—though the pattern it generates still has a simple repetitive form.

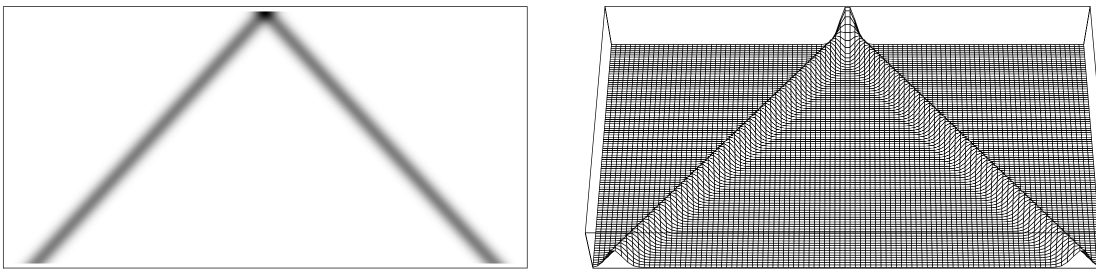
Considering the amount of mathematical work that has been done on partial differential equations, one might have thought that a vast range of different equations would by now have been studied. But in fact almost all the work—at least in one dimension—has concentrated on just the three specific equations on the facing page, together with a few others that are essentially equivalent to them.

And as we have seen, these equations yield only simple behavior.

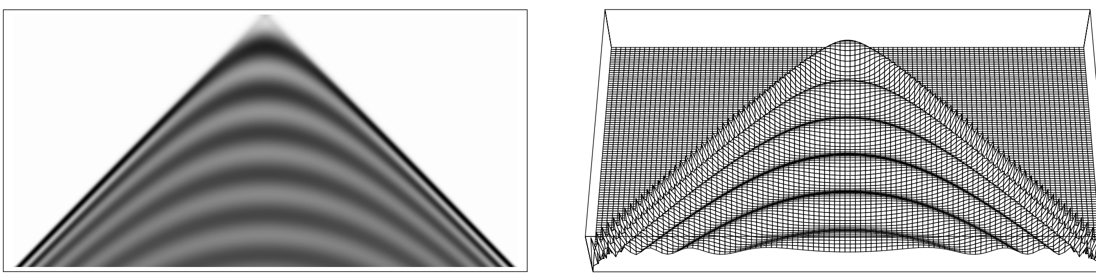
So is it in fact possible to get more complicated behavior in partial differential equations? The results in this book on other kinds of systems strongly suggest that it should be. But traditional



diffusion equation: $\partial_t u[t, x] = 1/4\, \partial_{xx} u[t, x]$



wave equation: $\partial_{tt} u[t, x] = \partial_{xx} u[t, x]$



sine-Gordon soliton equation: $\partial_{tt} u[t, x] = \partial_{xx} u[t, x] + \mathrm{Sin}[u[t, x]]$

Three partial differential equations that have historically been studied extensively. Just like in other pictures in this book, position goes across the page, and time down the page. In each equation u is the gray level at a particular point, $\partial_t u$ is the rate of change (derivative) of the gray level with time, and $\partial_{tt} u$ is the rate of change of that rate of change (second derivative). Similarly, $\partial_x u$ is the rate of change with position in space, and $\partial_{xx} u$ is the rate of change of that rate of change. On this page and the ones that follow the initial conditions used are $u = e^{-x^2}$, $\partial_t u = 0$.

mathematical methods give very little guidance about how to find such behavior. Indeed, it seems that the best approach is essentially just to search through many different partial differential equations, looking for ones that turn out to show complex behavior.

But an immediate difficulty is that there is no obvious way to sample possible partial differential equations. In discrete systems such as cellular automata there are always a discrete set of possible rules. But in partial differential equations any mathematical formula can appear.

Nevertheless, by representing formulas as symbolic expressions with discrete sets of possible components, one can devise at least some schemes for sampling partial differential equations.

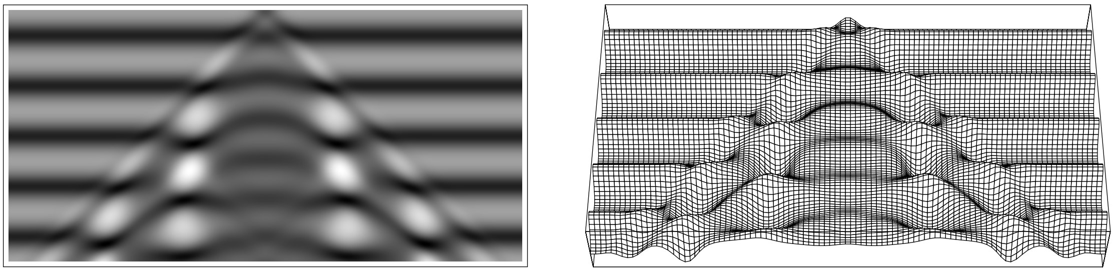
But even given a particular partial differential equation, there is no guarantee that the equation will yield self-consistent results. Indeed, for a very large fraction of randomly chosen partial differential equations what one finds is that after just a small amount of time, the gray level one gets either becomes infinitely large or starts to vary infinitely quickly in space or time. And whenever such phenomena occur, the original equation can no longer be used to determine future behavior.

But despite these difficulties I was eventually able to find the partial differential equations shown on the next two pages.

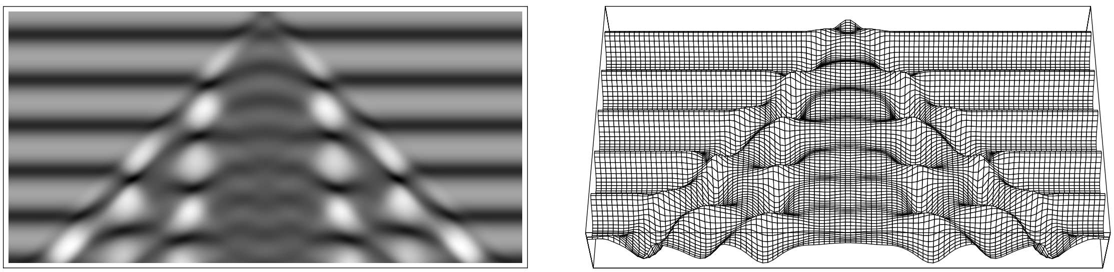
The mathematical statement of these equations is fairly simple. But as the pictures show, their behavior is highly complex.

Indeed, strangely enough, even though the underlying equations are continuous, the patterns they produce seem to involve patches that have a somewhat discrete structure.

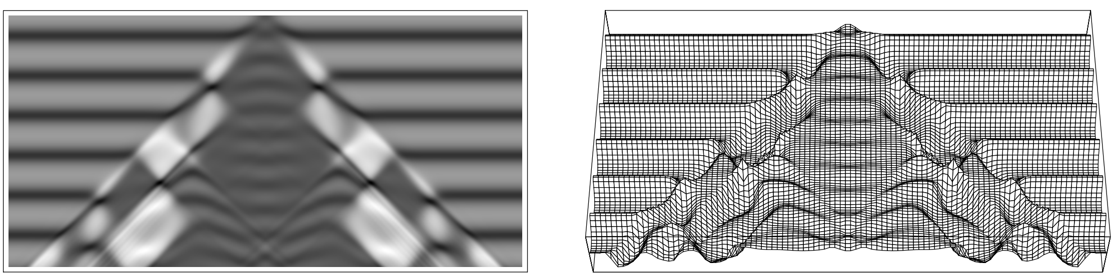
But the main point that the pictures on the next two pages make is that the kind of complex behavior that we have seen in this book is in no way restricted to systems that are based on discrete elements. It is certainly much easier to find and to study such behavior in these discrete systems, but from what we have learned in this section, we now know that the same kind of behavior can also occur in completely continuous systems such as partial differential equations.



$\partial_{tt}u[t, x] = \partial_{xx}u[t, x] + (1 - u[t, x]^2)(1 + u[t, x])$

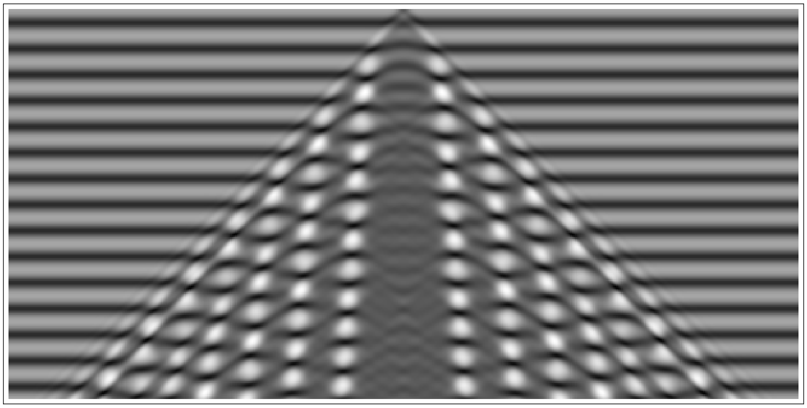


$\partial_{tt}u[t, x] = \partial_{xx}u[t, x] + (1 - u[t, x]^2)(1 + 2u[t, x])$

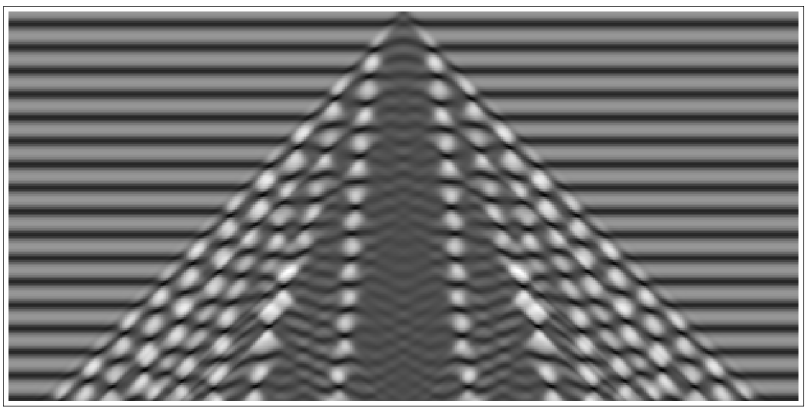


$\partial_{tt}u[t, x] = \partial_{xx}u[t, x] + (1 - u[t, x]^2)(1 + 4u[t, x])$

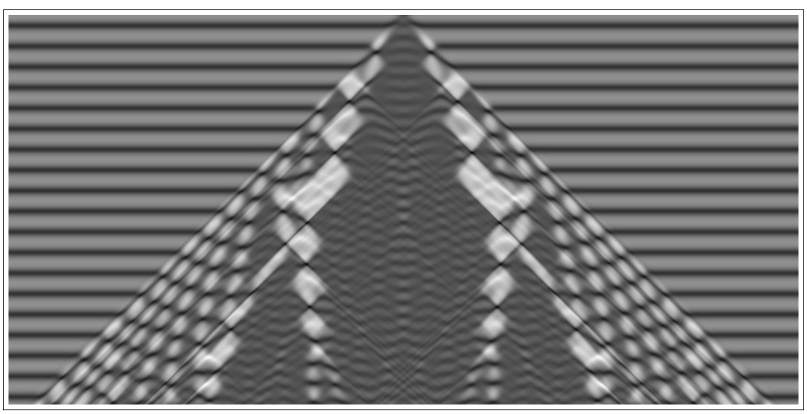
Examples of partial differential equations I have found that have more complicated behavior. The background in each case purely is repetitive, but the main part of the pattern is complex, and reminiscent of what is produced by continuous cellular automata and many other kinds of systems discussed in this book.



$\partial_{tt} u[t, x] = \partial_{xx} u[t, x] + (1 - u[t, x]^2)(1 + u[t, x])$



$\partial_{tt} u[t, x] = \partial_{xx} u[t, x] + (1 - u[t, x]^2)(1 + 2u[t, x])$



$\partial_{tt} u[t, x] = \partial_{xx} u[t, x] + (1 - u[t, x]^2)(1 + 4u[t, x])$