EXCERPTED FROM

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A NEW KIND OF SCIENCE

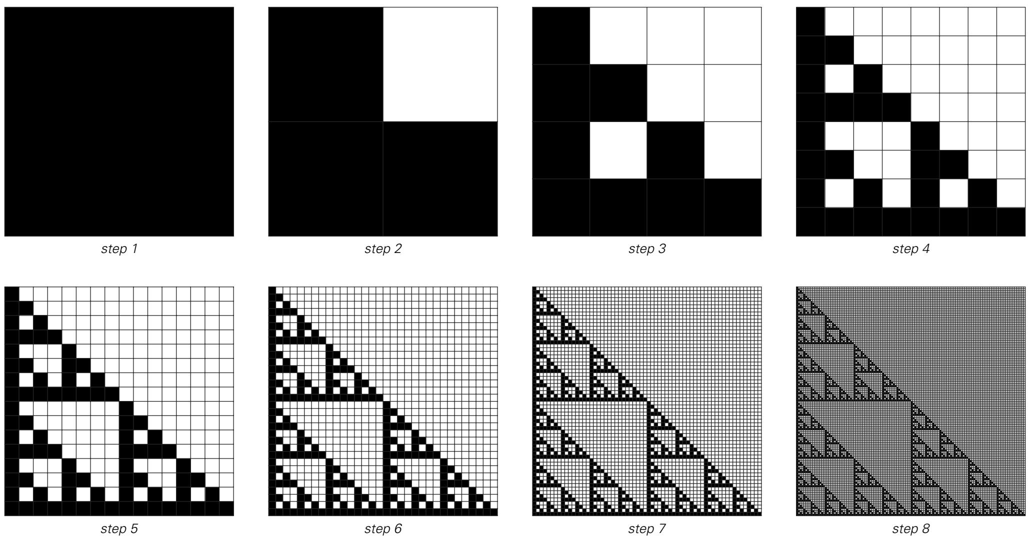
SECTION 5.4

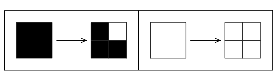
Substitution Systems and Fractals

Substitution Systems and Fractals

One-dimensional substitution systems of the kind we discussed on page 82 can be thought of as working by progressively subdividing each element they contain into several smaller elements.

One can construct two-dimensional substitution systems that work in essentially the same way, as shown in the pictures below.

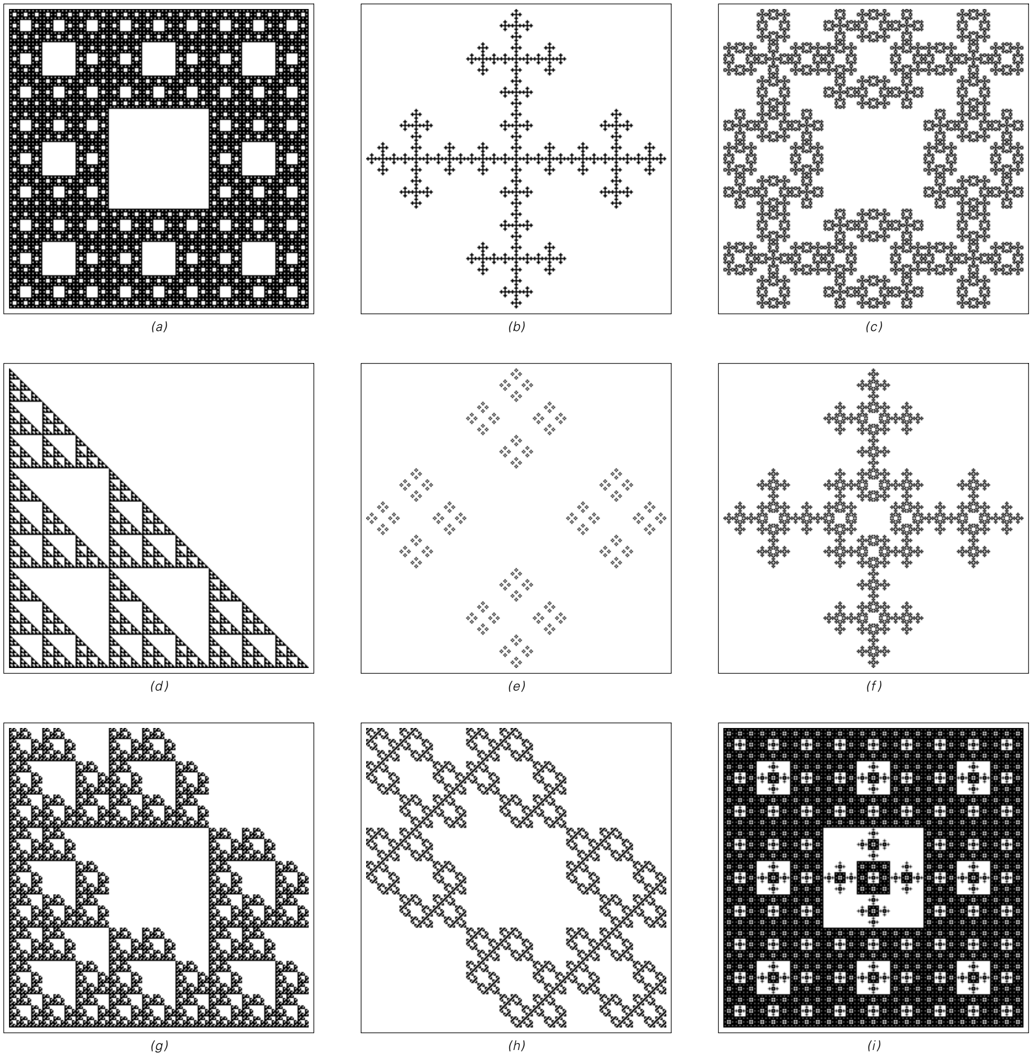




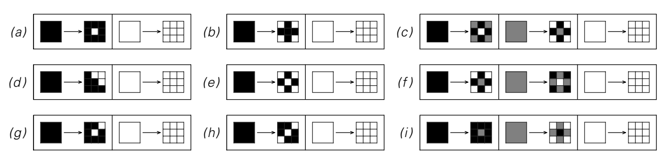
A two-dimensional substitution system in which each square is replaced by four smaller squares at every step according to the rule shown on the left. The pattern generated has a nested form.

The next page gives some more examples of two-dimensional substitution systems. The patterns that are produced are certainly quite intricate. But there is nevertheless great regularity in their overall forms. Indeed, just like patterns produced by one-dimensional substitution systems on page 83, all the patterns shown here ultimately have a simple nested structure.

Why does such nesting occur? The basic reason is that at every step the rules for the substitution system simply replace each black square with several smaller black squares. And on subsequent steps, each of these new black squares is then in turn replaced in exactly the

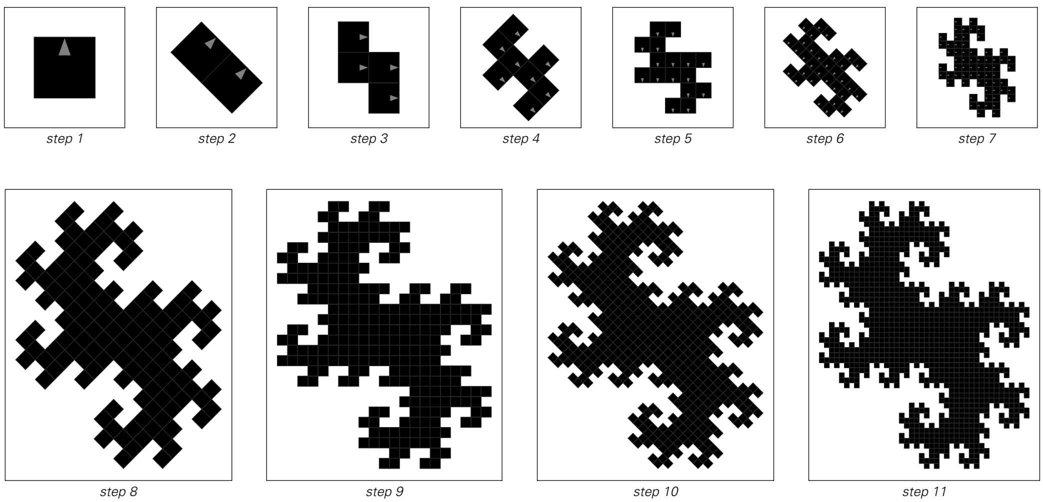


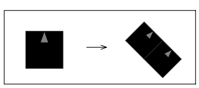
Patterns from various two-dimensional substitution systems. In each case what is shown is the pattern obtained after five steps of evolution according to the rules on the right, starting with a single black square.



same way, so that it ultimately evolves to produce an identical copy of the whole pattern.

But in fact there is nothing about this basic process that depends on the squares being arranged in any kind of rigid grid. And the picture below shows what happens if one just uses a simple geometrical rule to replace each black square by two smaller black squares. The result, once again, is that one gets an intricate but highly regular nested pattern.

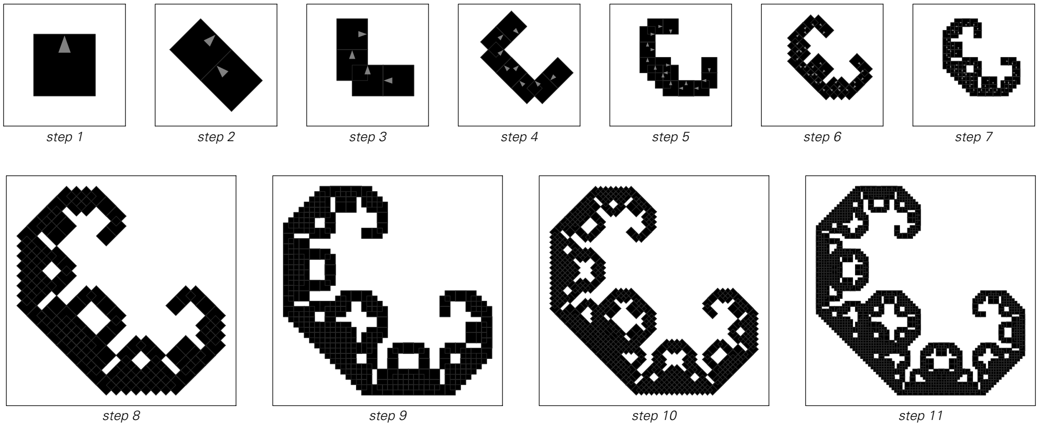


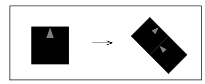


The pattern obtained by starting with a single black square and then at every step replacing each black cell with two smaller black cells according to the simple geometrical rule shown on the left. Note that in applying the rule to a particular square, one must take account of the orientation of that square. The final pattern obtained has an intricate nested structure.

In a substitution system where black squares are arranged on a grid, one can be sure that different squares will never overlap. But if there is just a geometrical rule that is used to replace each black square, then it is possible for the squares produced to overlap, as in the picture on the next page. Yet at least in this example, the overall pattern that is ultimately obtained still has a purely nested structure.

The general idea of building up patterns by repeatedly applying geometrical rules is at the heart of so-called fractal geometry. And the





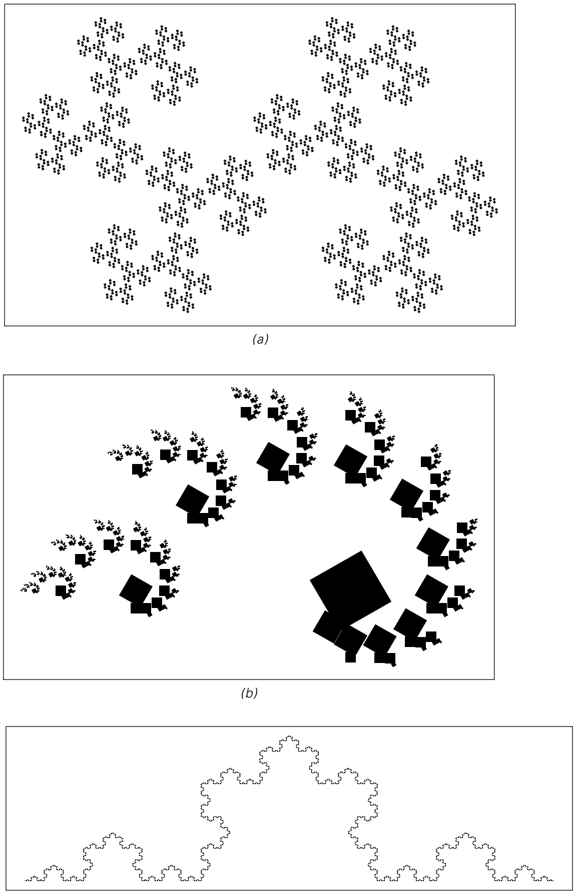
The pattern obtained by repeatedly applying the simple geometrical rule shown on the right. Even though this basic rule does not involve overlapping squares, the pattern obtained even by step 3 already has squares that overlap. But the overall pattern obtained after a large number of steps still has a nested form.

pictures on the facing page show several more examples of fractal patterns produced in this way.

The details of the geometrical rules used are different in each case. But what all the rules have in common is that they involve replacing one black square by two or more smaller black squares. And with this kind of setup, it is ultimately inevitable that all the patterns produced must have a completely regular nested structure.

So what does it take to get patterns with more complicated structure? The basic answer, much as we saw in one-dimensional substitution systems on page 85, is some form of interaction between different elements—so that the replacement for a particular element at a given step can depend not only on the characteristics of that element itself, but also on the characteristics of other neighboring elements.

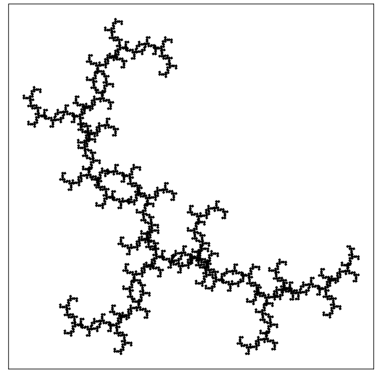
But with geometrical replacement rules of the kind shown on the facing page there is a problem with this. For elements can end up anywhere in the plane, making it difficult to define an obvious notion of neighbors. And the result of this has been that in traditional fractal geometry the idea of interaction between elements is not considered—so that all patterns that are produced have a purely nested form.



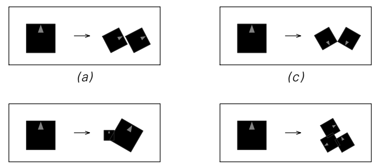
(a)

(b)

(c)



(d)



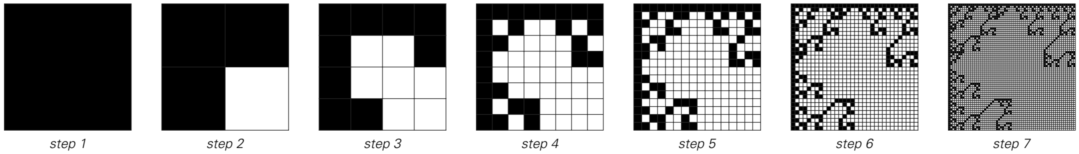
(a) (c)

(b) (d)

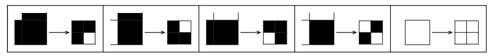
Examples of fractal patterns produced by repeatedly applying the geometrical rules shown for a total of 12 steps. The details of each pattern are different, but in all cases the patterns have a nested overall structure. The presence of this nested structure is an inevitable consequence of the fact that the rule for replacing an element at a particular position does not depend in any way on other elements.

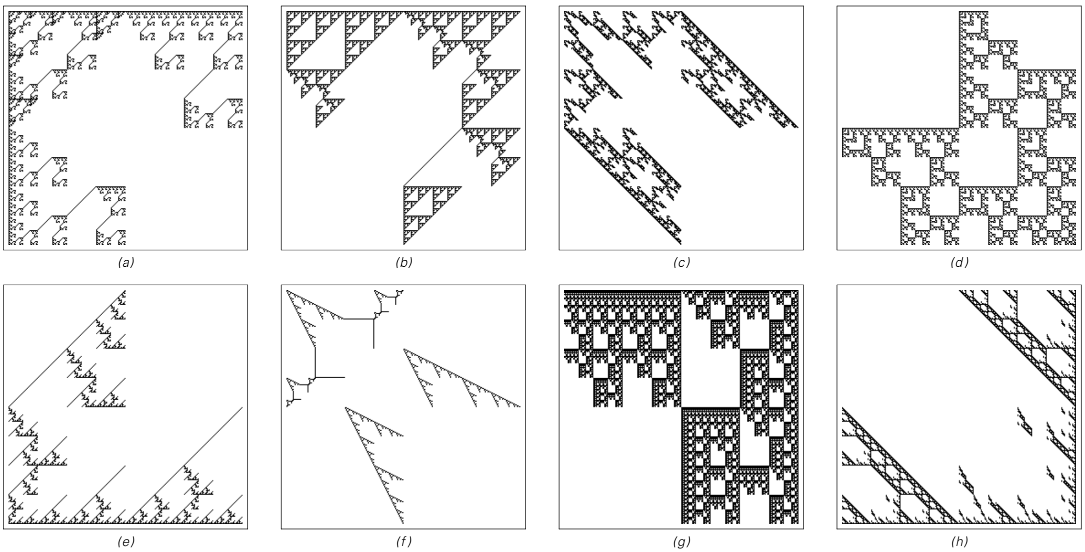
Yet if one sets up elements on a grid it is straightforward to allow the replacements for a given element to depend on its neighbors, as in the picture at the top of the next page. And if one does this, one immediately gets all sorts of fairly complicated patterns that are often not just purely nested—as illustrated in the pictures on the next page.

In Chapter 3 we discussed both ordinary one-dimensional substitution systems, in which every element is replaced at each step, and sequential substitution systems, in which just a single block of elements are replaced at each step. And what we did to find which block of elements should be replaced at a given step was to scan the whole sequence of elements from left to right.

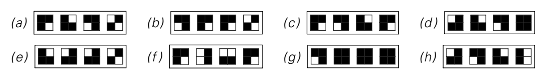


A two-dimensional neighbor-dependent substitution system. The grid of cells is assumed to wrap around in both its dimensions.





Patterns generated by 8 steps of evolution in various two-dimensional neighbor-dependent substitution systems.



So how can this be generalized to higher dimensions? On a two-dimensional grid one can certainly imagine snaking backwards and forwards or spiralling outwards to scan all the elements. But as soon as one defines any particular order for elements—however they may be laid out—this in effect reduces one to dealing with a one-dimensional system.

And indeed there seems to be no immediate way to generalize sequential substitution systems to two or more dimensions. In Chapter 9, however, we will see that with more sophisticated ideas it is in fact possible in any number of dimensions to set up substitution systems in which elements are scanned in order—but whatever order is used, the results are in some sense always the same.