EXCERPTED FROM

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A NEW KIND OF SCIENCE

SECTION 6.4

Systems of Limited Size and Class 2 Behavior

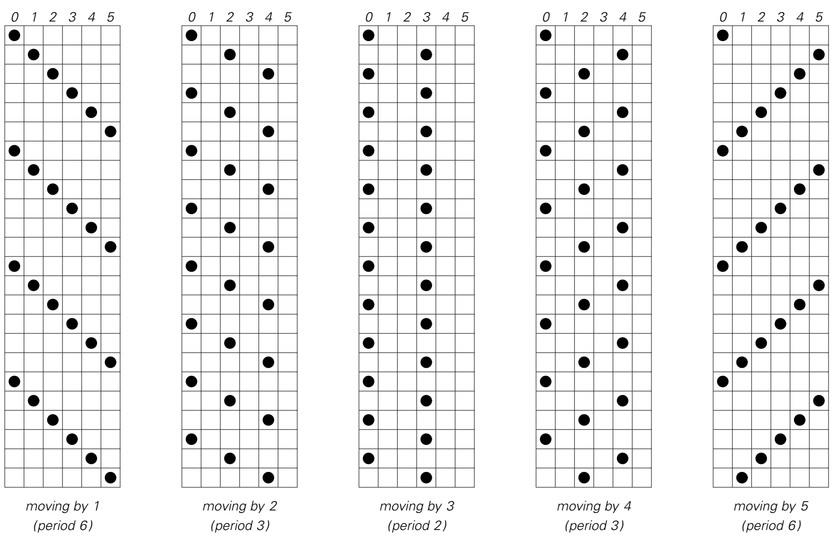
Systems of Limited Size and Class 2 Behavior

In the past two sections we have seen two important features of class 2 systems: first, that their behavior is always eventually repetitive, and second, that they do not support any kind of long-range communication.

So what is the connection between these two features?

The answer is that the absence of long-range communication effectively forces each part of a class 2 system to behave as if it were a system of limited size. And it is then a general result that any system of limited size that involves discrete elements and follows definite rules must always eventually exhibit repetitive behavior. Indeed, as we will discuss in the next chapter, it is this phenomenon that is ultimately responsible for much of the repetitive behavior that we see in nature.

The pictures below show a very simple example of the basic phenomenon. In each case there is a dot that can be in one of six possible positions. And at every step the dot moves a fixed number of positions to the right, wrapping around as soon as it reaches the right-hand end.

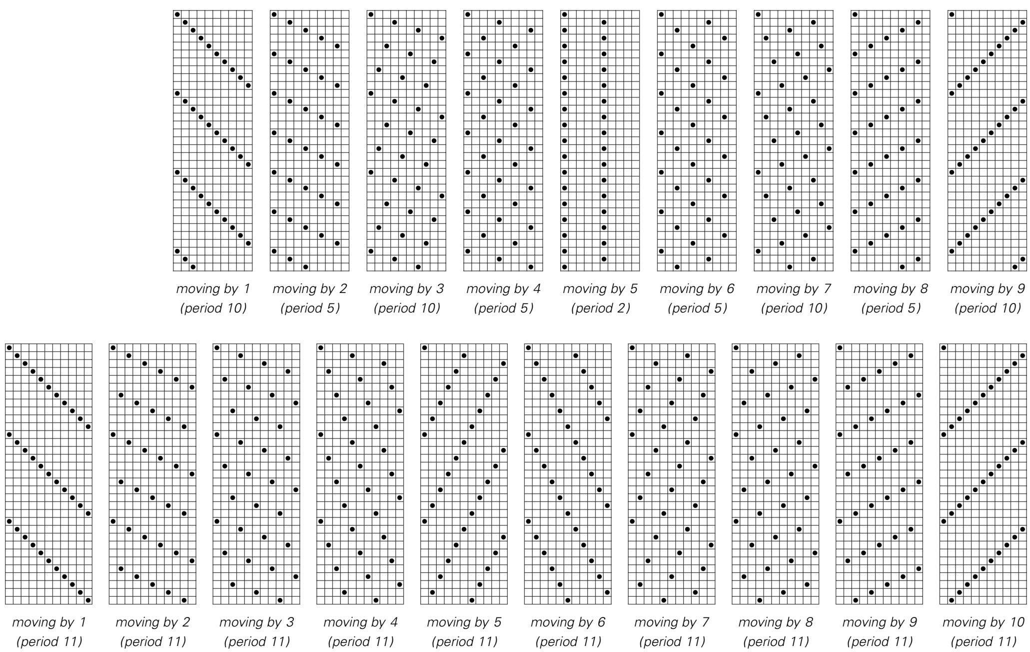


A simple system that contains a single dot which can be in one of six possible positions. At each step, the dot moves some number of positions to the right, wrapping around as soon as it reaches the right-hand end. The behavior of this system, like other systems of limited size, is always repetitive.

Looking at the pictures we then see that the behavior which results is always purely repetitive—though the period of repetition is different in different cases. And the basic reason for the repetitive behavior is that whenever the dot ends up in a particular position, it must always repeat whatever it did when it was last in that position.

But since there are only six possible positions in all, it is inevitable that after at most six steps the dot will always get to a position where it has been before. And this means that the behavior must repeat with a period of at most six steps.

The pictures below show more examples of the same setup, where now the number of possible positions is 10 and 11. In all cases, the behavior is repetitive, and the maximum repetition period is equal to the number of possible positions.

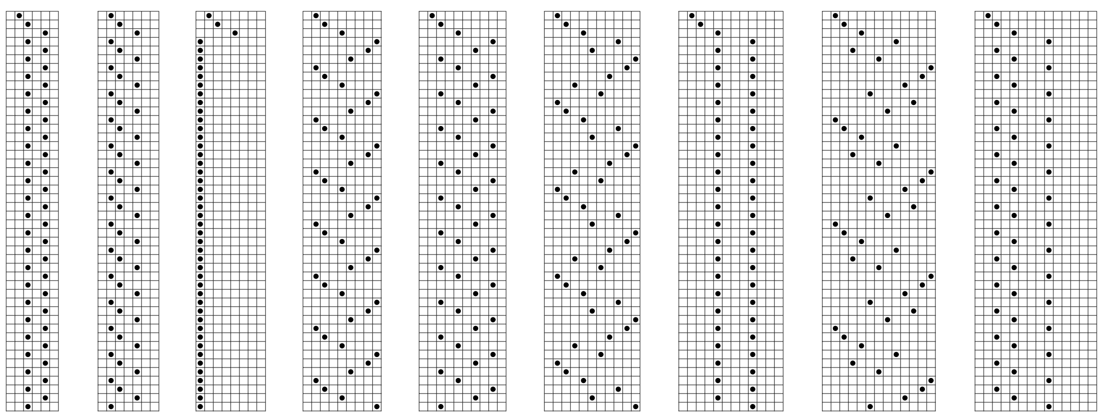


More examples of the type of system shown on the previous page, but now with 10 and 11 possible positions for the dot. The behavior always repeats itself in at most 10 or 11 steps. But the exact number of steps in each case depends on the prime factors of the numbers that define the system.

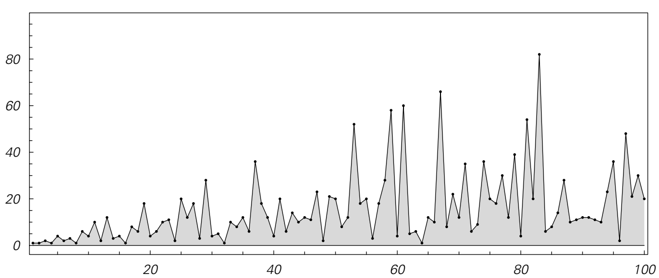
Sometimes the actual repetition period is equal to this maximum value. But often it is smaller. And indeed it is a common feature of systems of limited size that the repetition period one sees can depend greatly on the exact size of the system and the exact rule that it follows.

In the type of system shown on the facing page, it turns out that the repetition period is maximal whenever the number of positions moved at each step shares no common factor with the total number of possible positions—and this is achieved for example whenever either of these quantities is a prime number.

The pictures below show another example of a system of limited size based on a simple rule. The particular rule is at each step to double the number that represents the position of the dot, wrapping around as soon as this goes past the right-hand end.



size 6 (period 2) | size 7 (period 3) | size 8 (period 1) | size 9 (period 6) | size 10 (period 4) | size 11 (period 10) | size 12 (period 2) | size 13 (period 12) | size 14 (period 3)



A system where the number that represents the position of the dot doubles at each step, wrapping around whenever it reaches the right-hand end. (After t steps the dot is thus at position *Mod[*2^t*, n]* in a size n system.) The plot at left gives the repetition period for this system as a function of its size; for odd n this period is equal to *MultiplicativeOrder[2, n]*.

Once again, the behavior that results is always repetitive, and the repetition period can never be greater than the total number of possible positions for the dot. But as the picture shows, the actual repetition period jumps around considerably as the size of the system is changed. And as it turns out, the repetition period is again related to the factors of the number of possible positions for the dot—and tends to be maximal in those cases where this number is prime.

So what happens in systems like cellular automata?

The pictures on the facing page show some examples of cellular automata that have a limited number of cells. In each case the cells are in effect arranged around a circle, so that the right neighbor of the rightmost cell is the leftmost cell and vice versa.

And once again, the behavior of these systems is ultimately repetitive. But the period of repetition is often quite large.

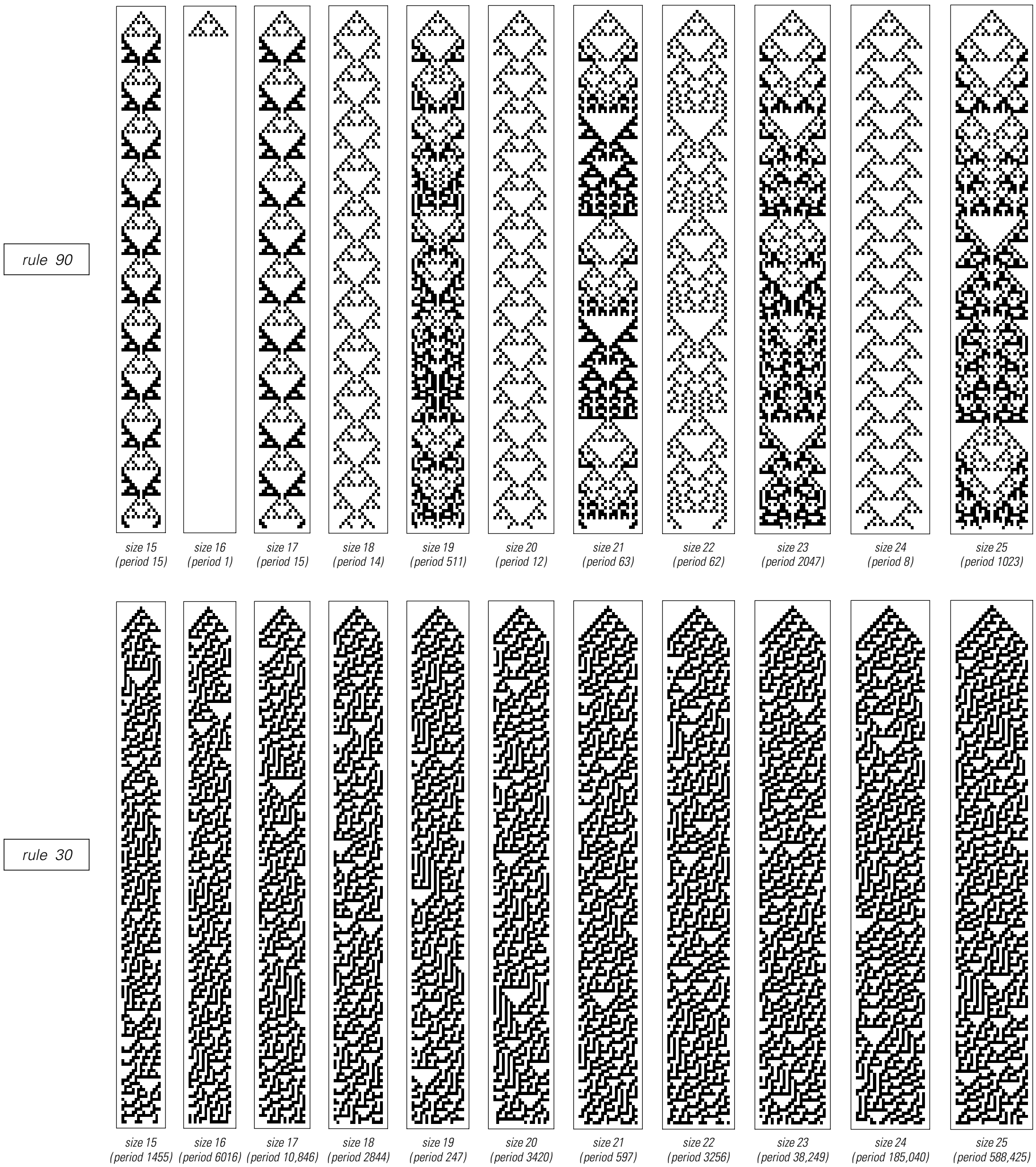
The maximum possible repetition period for any system is always equal to the total number of possible states of the system.

For the systems involving a single dot that we discussed above, the possible states correspond just to possible positions for the dot, and the number of states is therefore equal to the size of the system.

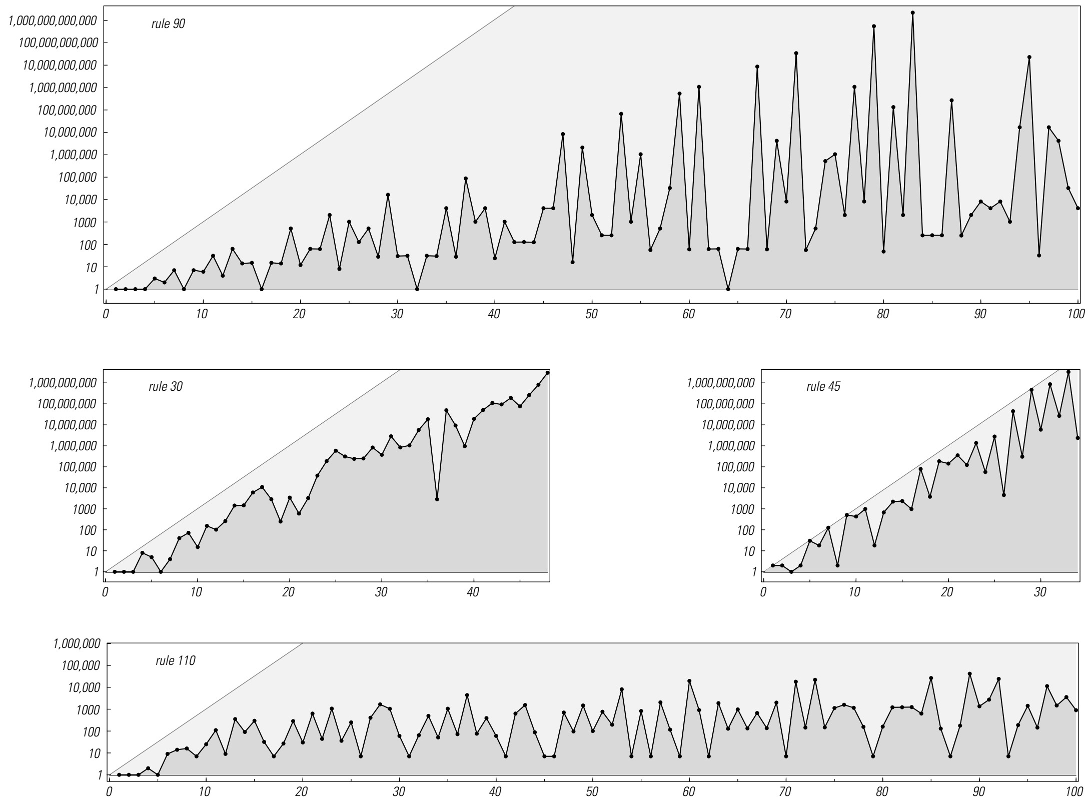
But in a cellular automaton, every possible arrangement of black and white cells corresponds to a possible state of the system. With n cells there are thus 2^n possible states. And this number increases very rapidly with the size n: for 5 cells there are already 32 states, for 10 cells 1024 states, for 20 cells 1,048,576 states, and for 30 cells 1,073,741,824 states.

The pictures on the next page show the actual repetition periods for various cellular automata. In general, a rapid increase with size is characteristic of class 3 behavior. Of the elementary rules, however, only rule 45 seems to yield periods that always stay close to the maximum of 2^n. And in all cases, there are considerable fluctuations in the periods that occur as the size changes.

So how does all of this relate to class 2 behavior? In the examples we have just discussed, we have explicitly set up systems that have limited size. But even when a system in principle contains an infinite number of cells it is still possible that a particular pattern in that system will only grow to occupy a limited number of cells. And in any



The behavior of cellular automata with a limited number of cells. In each case the right neighbor of the rightmost cell is taken to be the leftmost cell and vice versa. The pattern produced always eventually repeats, but the period of repetition can increase rapidly with the size of the system.



Repetition periods for various cellular automata as a function of size. The initial conditions used in each case consist of a single black cell, as in the pictures on the previous page. The dashed gray line indicates the maximum possible repetition period of 2^n. The maximum repetition period for rule 90 is $2^{(n-1)/2} - 1$. For rule 30, the peak repetition periods are of order $2^{0.63 n}$, while for rule 45, they are close to 2^n (for $n = 29$, for example, the period is 463,347,935, which is 86% of the maximum possible). For rule 110, the peaks seem to increase roughly like n^3.

such case, the pattern must repeat itself with a period of at most 2^n steps, where n is the size of the pattern.

In a class 2 system with random initial conditions, a similar thing happens: since different parts of the system do not communicate with each other, they all behave like separate patterns of limited size. And in fact in most class 2 cellular automata these patterns are effectively only a few cells across, so that their repetition periods are necessarily quite short.