EXCERPTED FROM

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A NEW KIND OF SCIENCE

SECTION 9.7

Space as a Network

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In the last section I argued that if the ultimate model of physics is to be as simple as possible, then one should expect that all the features of our universe must at some level emerge purely from properties of space. But what should space be like if this is going to be the case?

The discussion in the section before last suggests that for the richest properties to emerge there should in a sense be as little rigid underlying structure built in as possible. And with this in mind I believe that what is by far the most likely is that at the lowest level space is in effect a giant network of nodes.

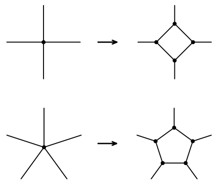
In an array of cells like in a cellular automaton each cell is always assigned some definite position. But in a network of nodes, the nodes are not intrinsically assigned any position. And indeed, the only thing that is defined about each node is what other nodes it is connected to.

Yet despite this rather abstract setup, we will see that with a sufficiently large number of nodes it is possible for the familiar properties of space to emerge—together with other phenomena seen in physics.

I already introduced in Chapter 5 a particular type of network in which each node has exactly two outgoing connections to other nodes, together with any number of incoming connections. The reason I chose this kind of network in Chapter 5 is that there happens to be a fairly easy way to set up evolution rules for such networks. But in trying to find an ultimate model of space, it seems best to start by considering networks that are somehow as simple as possible in basic structure—and it turns out that the networks of Chapter 5 are somewhat more complicated than is necessary.

For one thing, there is no need to distinguish between incoming and outgoing connections, or indeed to associate any direction with each connection. And in addition, nothing fundamental is lost by requiring that all the nodes in a network have exactly the same total number of connections to other nodes.

With two connections, only very trivial networks can ever be made. But if one uses three connections, a vast range of networks immediately become possible. One might think that one could get a

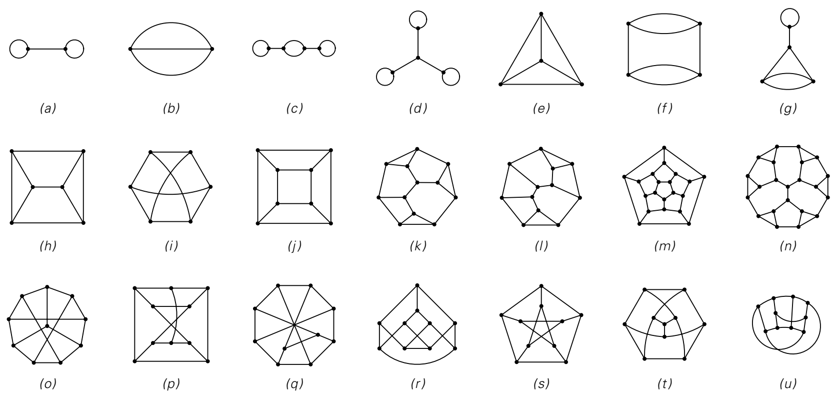


Examples of how nodes with more than three connections can be decomposed into collections of nodes with exactly three connections.

fundamentally larger range if one allowed, say, four or five connections rather than just three. But in fact one cannot, since any node with more than three connections can in effect always be broken into a collection of nodes with exactly three connections, as in the pictures on the left.

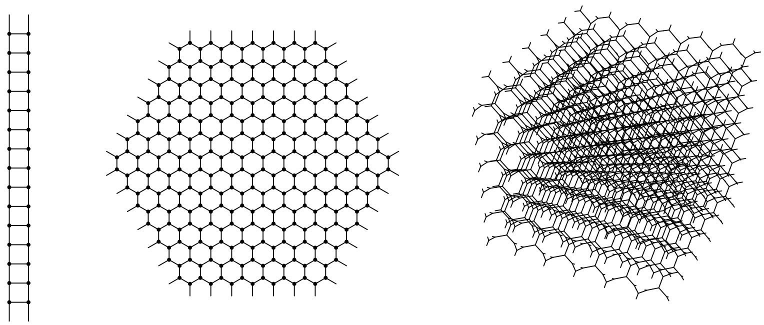
So what this means is that it is in a sense always sufficient to consider networks with exactly three connections at each node. And it is therefore these networks that I will use here in discussing fundamental models of space.

The pictures below show a few small examples of such networks. And already considerable diversity is evident. But none of the networks shown seem to have many properties familiar from ordinary space.



Examples of small networks with exactly three connections at each node. The first line shows all possible networks with up to four nodes. In what follows I consider only non-degenerate networks, in which there is at most one connection between any two nodes. Example (i) is the smallest network that cannot be drawn in two dimensions without lines crossing. Examples (k) and (l) are the smallest networks that have no symmetries between different nodes. Example (e) corresponds to the net of a tetrahedron, (j) to the net of a cube, and (m) to the net of a dodecahedron. Examples (o) through (u) show seven ways of drawing the same network, in this case the so-called Petersen network.

So how then can one get networks that correspond to ordinary space? The first step is to consider networks that have much larger numbers of nodes. And as examples of these, the pictures at the top of the facing page show networks that are specifically constructed to correspond to ordinary one-, two- and three-dimensional space.

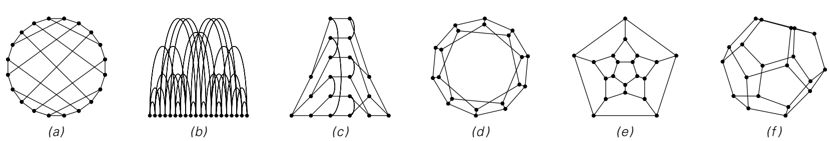


Examples of networks with three connections at each node that are effectively one, two and three-dimensional. These networks can be continued forever, and all have the property of being homogeneous, in the sense that every node has an environment identical to every other node.

Each of these networks is at the lowest level just a collection of nodes with certain connections. But the point is that the overall pattern of these connections is such that on a large scale there emerges a clear correspondence to ordinary space of a particular dimension.

The pictures above are drawn so as to make this correspondence obvious. But what if one was just presented with the raw pattern of connections for some network? How could one see whether the network could correspond to ordinary space of a particular dimension?

The pictures below illustrate the main difficulty: given only its pattern of connections, a particular network can be laid out in many completely different ways, most of which tell one very little about its potential correspondence with ordinary space.



Six different ways of laying out the same network. (a) nodes arranged around a circle; (b) nodes arranged along a line; (c) nodes arranged across the page according to distance from a particular node; (d) 2D layout with network and spatial distances as close as possible; (e) planar layout; (f) 3D layout.

So how then can one proceed? The fundamental idea is to look at properties of networks that can both readily be deduced from their pattern of connections and can also be identified, at least in some large-scale limit, with properties of ordinary space. And the notion of distance is perhaps the most fundamental of such properties.

A simple way to define the distance between two points is to say that it is the length of the shortest path between them. And in ordinary space, this is normally calculated by subtracting the numerical coordinates of the positions of the points. But on a network things become more direct, and the distance between two nodes can be taken to be simply the minimum number of connections that one has to follow in order to get from one node to the other.

But can one tell just by looking at such distances whether a particular network corresponds to ordinary space of a certain dimension?

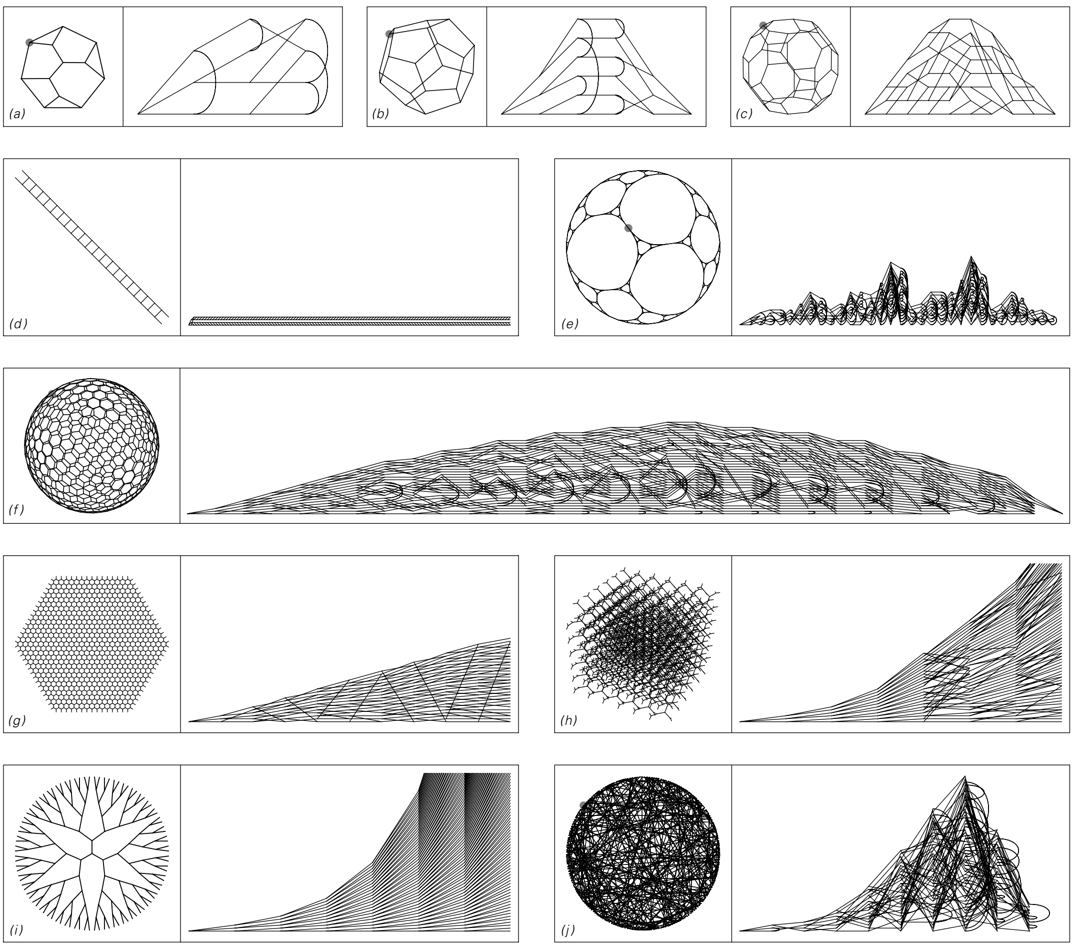
To a large extent one can. And a test is to see whether there is a way to lay out the nodes in the network in ordinary space so that the distances between nodes computed from their positions in space agree—at least in some approximation—with the distances computed directly by following connections in the network.

The three networks at the top of the previous page were laid out precisely so as to make this the case respectively for one, two and three-dimensional space. But why for example can the second network not be laid out equally well in one-dimensional rather than two-dimensional space? One way to see this is to count the number of nodes that appear at a given distance from a particular node in the network.

And for this specific network, the answer for this is very simple: at distance r there are exactly $3r$ nodes—so that the total number of nodes out to distance r grows like r^2. But now if one tried to lay out all these nodes in one dimension it is inevitable that the network would have to bulge out in order to fit in all the nodes. And it turns out that it is uniquely in two dimensions that this particular network can be laid out in a regular way so that distances based on following connections in it agree with ordinary distances in space.

For the other two networks at the top of the previous page similar arguments can be given. And in fact in general the condition for a network to correspond to ordinary d-dimensional space is precisely that the total number of nodes that appear in it out to distance r grows in some limiting sense like r^d—a result analogous to the standard mathematical fact that the area of a two-dimensional circle is πr^2, while the volume of a three-dimensional sphere is $4/3\,\pi\,r^3$, the volume of a four-dimensional hypersphere is $1/2\,\pi^2\,r^4$, and so on.

Below I show pictures of various networks. In each case the first picture is drawn to emphasize obvious regularities in the network. But the second picture is drawn in a more systematic way—by picking a specific starting node, and then laying out other nodes so that those at



Examples of various networks, shown first to emphasize their regularities, and second to illustrate the number of nodes reached by going successively more steps from a given node. For networks that in a limiting sense correspond to ordinary d-dimensional space, this number grows like r^{d-1}. All the larger networks shown are approximately uniform, in the sense that similar results are obtained starting from any node. Network (e) effectively has limiting dimension *Log[2, 3]* $\simeq$ *1.58*.

successively greater network distances appear in successive columns across the page. And this setup has the feature that the height of column r gives the number of nodes that are at network distance r.

So by looking at how these heights grow across the page, one can see whether there is a correspondence with the r^{d-1} form that one expects for ordinary d-dimensional space. And indeed in case (g), for example, one sees exactly r^1 linear growth, reflecting dimension 2.

Similarly, in case (d) one sees r^0 growth, reflecting dimension 1, while in case (h) one sees r^2 growth, reflecting dimension 3.

Case (f) illustrates slightly more complicated behavior. The basic network in this case locally has an essentially two-dimensional form—but at large scales it is curved by being wrapped around a sphere. And what therefore happens is that for fairly small r one sees r^1 growth—reflecting the local two-dimensional form—but then for larger r there is slower growth, reflecting the presence of curvature.

Later in this chapter we will see how such curvature is related to the phenomenon of gravity. But for now the point is just that network (f) again behaves very much like ordinary space with a definite dimension.

So do all sufficiently large networks somehow correspond to ordinary space in a certain number of dimensions? The answer is definitely no. And as an example, network (i) from the previous page has a tree-like structure with 3^r nodes at distance r. But this number grows faster than r^d for any d—implying that the network has no correspondence to ordinary space in any finite number of dimensions.

If the connections in a network are chosen at random—as in case (j)—then again there will almost never be the kind of locality that is needed to get something that corresponds to ordinary finite-dimensional space.

So what might an actual network for space in our universe be like?

It will certainly not be as simple and regular as most of the networks on the previous page. For within its pattern of connections must be encoded everything we see in our universe.

And so at the level of individual connections, the network will most likely at first look quite random. But on a larger scale, it must be arranged so as to correspond to ordinary three-dimensional space. And somehow whatever rules update the network must preserve this feature.